

Low-dimensional modelling of dynamical systems

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21 February, 1997

Abstract

Consider briefly the equations of fluid dynamics—they describe the enormous wealth of detail in all the interacting physical elements of a fluid flow—whereas in applications we want to deal with a description of just that which is interesting. In a wide variety of situations, simple approximate models are needed to perform practical simulations and make forecasts. I review the *derivation*, from a mathematical description of the detailed dynamics, of accurate, complete and useful low-dimensional models of the interesting dynamics in a system. The development of centre manifold theory and associated techniques puts this modelling process on a firm basis. As in Guckenheimer & Holmes [79, §2.5]:

“...these new methods will really be conventional perturbation style analyses interpreted geometrically...”

But the geometric viewpoint of dynamical systems theory greatly enriches our approach by providing a rationale for also deriving correct initial conditions, forcing and boundary conditions for the models—all essential elements of a model.

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1 Introduction

In secondary school we learn of the parabolic flight of a ball. In university physics we learn how it spins. In elasticity courses we learn that a ball deforms as it spins and bounces. In each successive stage of the modelling of the ball we deal with more details of the dynamics: first the position and velocity of the centre of mass, then position, orientation and their velocities, and lastly the position and velocity of every part of the ball. *In this paper I review the reverse of this process:* namely the reduction from equations describing the very detailed dynamics of a system down to simpler “model” equations dealing with the dynamics at a relatively coarse level of description.

A model is “simple” if it deals with just a few characteristics of the physical problem. For example, in the flight of the ball we describe just the movement of the centre of mass rather than the movement of each part of the ball. Describing just a few characteristics and so written in terms of just a few parameters, *a simple model is of low-dimension* when compared with the actual physical system. A model is useful if it describes the dynamics of interest with very little extraneous details: we are usually interested in how the ball as a whole moves without needing to know anything about its deformation as it spins and bounces. It is the rationale, theory and practice of such reduction in dimensionality that I review.

The general challenge is to start with an accurate and reliable description of the dynamics of a system of interest, a “Theory Of Everything” (TOE) as I wryly call it in this article, then to analyse it systematically and extract routinely the simple, low-dimensional dynamical models which are relevant in given situations:

$$\text{TOE} \longrightarrow \text{model}.$$

To do this, we invoke centre manifold theory, introduced in §2, and associated techniques based upon geometric pictures in the state space of the dynamical system. The use of centre manifold theory for this modelling was initiated by Coullet & Spiegel [44] and Carr & Muncaster [28, 27]. The rationale of exponential collapse that underpins centre manifold theory has been also invoked in other methods, but I argue that this approach has many advantages over the alternatives.

Applications of the theory are widespread, some are discussed in §3 and an historical perspective given by Coullet & Spiegel [44, §1]. Specific examples of this process of modelling by dimensional reduction are:

- elasticity \rightarrow rigid body motion (mentioned above);
- heat or mass transfer \rightarrow dispersion in a pipe or channel (§§3.2);
- Navier-Stokes equations \rightarrow thin films, sheets & jets of fluids (§§3.3);
- Navier-Stokes & heat \rightarrow convection (§§3.4);
- Markov chain \rightarrow quasi-stationary distribution [131];
- elasticity \rightarrow beam theory of bending and torsion (§§5.1);
- atmospheric models \rightarrow quasi-geostrophic approximation (§§5.2).

Although my interest, and thus this review, lies primarily in the field of fluid mechanics, the modelling issues addressed are relevant to any evolutionary system. However, there is not yet enough theory to support all the interesting applications of the concepts and techniques that we encounter. The main limitation on rigour is that in many of these applications the “low-dimensional” model is still actually of “infinite dimension”, a case for which there is almost no theory that is both rigorous and useful—more specific comments are sprinkled throughout.

There are many advantages in the centre manifold approach when compared with other methods of analysing dynamical systems to develop low-dimensional dynamical models.

Competing small effects need not appear at leading order in the analysis, §4; consequently you obtain the flexibility to justifiably adapt the model to different applications *without* redoing the whole analysis. Just one example of this flexibility is used to modify the governing equations of thin fluid films, §§3.3, so that the model incorporates the extra dynamical degree of freedom to resolve wave dynamics on the film. To recover a description of the physical problem, we sum to high-order in the modification while only considering adequate low-order physical terms. Such flexibility also justifies a local description of the interaction between counter-propagating waves.

The geometric picture of evolution near the centre manifold suggests analysis, §§6.1, that provides correct initial conditions to use with the model to make correct long-term forecasts. The algebra is based upon how neighbouring trajectories evolve, identifying which ones approach each other exponentially quickly and thus have the same long-term evolution. For example, in the long-term dispersion down a channel (§§3.2) we can discern the difference between dumping contaminant into the slow moving flow near the bank and into the fast core flow. Normal form transformations, §§6.2, also illustrate the projection of initial conditions onto the centre manifold. In addition,

normal form transformations show limitations in the so-called slow manifold models, §5, that are formed by removing the dynamics of fast oscillations (as in beam theory or rigid-body dynamics). Through nonlinear interaction, neglected fast waves may resonate and cause inevitable errors to accumulate over time. In contrast, centre manifold models guarantee the existence of forecasts that are accurate exponentially quick.

One attribute of basic centre manifold theory is that it deals with autonomous dynamical systems. However, in the presence of a time-dependent forcing, §§7.1, the system is pushed away from the centre manifold and so we use the geometric projection of initial conditions to determine what forcing is appropriate in the model. I show, for example, that a model may be very sensitive to a forcing which more primitive approaches would neglect. There is also many interesting issues in the modelling of noisy dynamical systems, as expressed by stochastic differential equations. I argue, §§7.2, that centre manifold concepts provide relatively straightforward tools to approach this problem rationally. These methods should apply to interesting questions such as: what influence may turbulence have on dispersion? and how does substrate roughness affect the flow of thin films?

Many useful models are expressed in terms of partial differential equations in space and time. Such partial differential equations must have boundary conditions. For example, models for dispersion in a channel or for beam theory require boundary conditions at the inlet and outlet of the channel or the ends of the beam respectively. Arguments based upon the *spatial evolution* away from the boundary, applied to both the full system and the model, give a rationale which provides correct boundary conditions, §8. To leading order these boundary conditions are typically those obtained from physical heuristics. However, there are corrections accounting for more subtle features of the dynamics.

Computing details of the centre manifold and its dynamic model often involves considerable algebra. This is especially true for centre manifolds of more than just a few dimensions as the algebraic complexity of the model may increase combinatorially. Computer algebra can be used to minimise the human labour involved. After all:

“It is unworthy of excellent persons to lose hours like slaves in the labour of calculation”... Gottfried Wilhelm von Leibniz.

The challenge addressed in §9 is to develop algorithms for computer based algebra packages which are simple and reliable to implement.

All the above aspects are features of a complete approach to modelling dynamical systems based upon centre manifold theory, techniques and concepts.

Throughout this review we focus on continuous time dynamics, flows, as expressed through differential equations. Similar concepts and analysis can be also developed for discrete maps but I will not elaborate on these.

2 Rational modelling theory

Simple models must have low dimension corresponding to the few variables in the model. Since a model must have something to say about the TOE—states of the model correspond to states of the TOE—the state space of the model must be able to be embedded in the high-dimensional state space of the TOE. Typically we imagine that the state space of a model forms a low-dimensional differentiable manifold within the state space of the TOE.

If the model is to accurately capture the dynamics of the TOE, then the manifold must be made of some of the trajectories of the TOE. The detail lost in apparently ignoring all the dynamics outside the states described by the model is an inevitable consequence of forming a simple, low-dimensional model. The process of analysing the TOE and creating a low-dimensional model is sometimes termed *coarse graining* [117, e.g.] because of the loss of fine detail in forming a model. In this section I introduce centre manifold theory as a basis for the rational modelling of dynamical systems, as first recognised by Coullet & Spiegel [44] and Carr & Muncaster [28, 27].

2.1 Exponential collapse gives a rationale

If many modes of the TOE decay exponentially, then all that is left after the transients decay are the *relatively* slowly evolving modes of long-term importance. The evolution of these few significant modes effectively forms a low-dimensional dynamical system on a low-dimensional set of states in state space. Through the rapid exponential decay all neighbouring trajectories are quickly attracted to these low-dimensional dynamics and so they form an accurate low-dimensional model of the TOE.

This idea also lies behind the construction of so-called *inertial manifolds* by Temam [171] and others. Functional analysis is used to construct inertial manifolds and estimate some properties such as their dimension (see Foias *et al* [67, 66] for example). But what if we are not just interested in the ultimate attractor, but are curious about long-lasting transients? After all, the eventual fate of the universe is either the big crunch or a featureless death by high entropy—but before these become an overriding issue to humans there are many problems of interest to study, albeit transient. A more prosaic example is the dispersion of material in an infinitely long channel or pipe: the

ultimate attractor is a completely dispersed contaminant of effectively zero concentration everywhere; however, as seen in §§3.2, we are very interested in modelling the long transient of how a contaminant spreads as it is carried downstream.

One case where the distinction between exponential decay and long-lasting importance can be made with absolute clarity is in the neighbourhood of an equilibrium or fixed point. Without loss of generality we take the reference equilibrium at the origin. Let the *linearised* dynamics be $\dot{\mathbf{u}} = \mathcal{L}\mathbf{u}$, then the eigenvalues of the linear operator \mathcal{L} determine the dynamics in the neighbourhood:

- modes with $\Re(\lambda) < 0$ decay exponentially;
- modes with $\Re(\lambda) = 0$ do not decay, are long-lasting and form the basis of a low-dimensional model.

An elementary example from [137] is

$$\dot{x} = -xy, \quad \text{and} \quad \dot{y} = -y + x^2 - 2y^2. \quad (1)$$

Very quickly all trajectories approach a curved “subspace” in state space, called the centre manifold, $y = x^2$ in this example as shown in Figure 1. Thus no matter what the initial condition, the only states of long-term interest are those of the centre manifold. The evolution on the centre manifold, here $\dot{x} = -x^3$, then forms a low-dimensional model of *all* the dynamics in the system.

In the above, I introduced the criterion that $\Re(\lambda) = 0$ for determining the modes that form the low-dimensional model. Arguments have been devised to relax this restrictive equality. An argument could be made that modes with $\Re(\lambda) < -\gamma < 0$ decay exponentially, whereas modes with $\Re(\lambda) \geq -\gamma$ are at least *longer*-lasting and form the basis of a low-dimensional model—here it is the modes with $\Re(\lambda) < -\gamma$ that “collapse” the state space. For an elementary example, also from [137], consider

$$\dot{x} = \epsilon x - xy, \quad \text{and} \quad \dot{y} = -y + x^2 - 2y^2, \quad (2)$$

which has the exponentially attractive manifold

$$y = \frac{x^2}{1 + 2\epsilon}, \quad (3)$$

as seen in Figure 2, on which the model evolution is $\dot{x} = \epsilon x - x^3/(1 + 2\epsilon)$. For $\epsilon \geq 0$ this is termed a *centre-unstable manifold*, a concept used by Armbruster

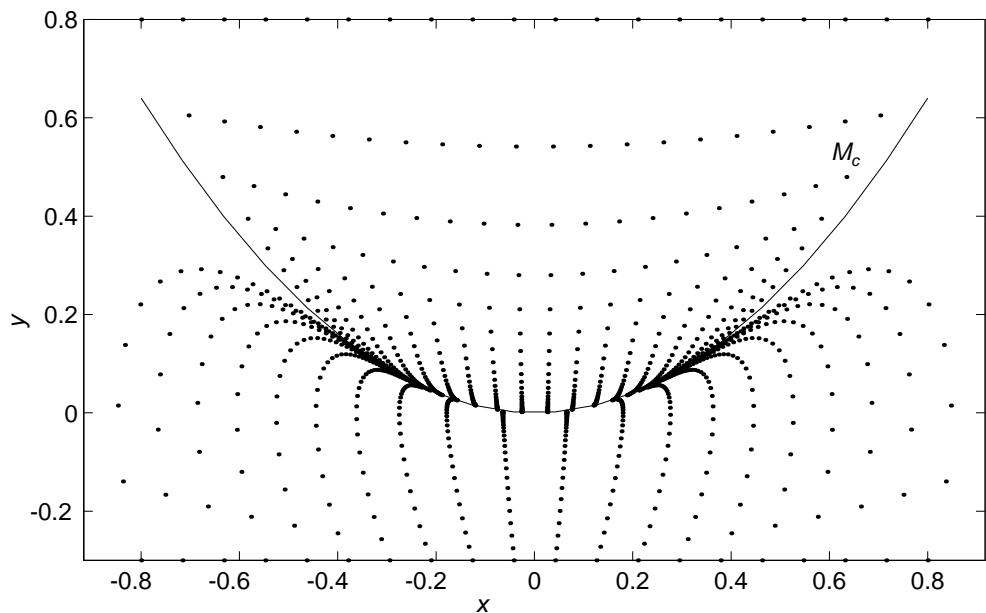


Figure 1: trajectories of the dynamical system (1) plotted every $\Delta t = 0.2$ to show the rapid approach, roughly in 2 units of time, to the centre manifold, \mathcal{M}_c . Observe how all these initial conditions collapse onto a one-dimensional set of states in which *all* the long-term dynamics take place.

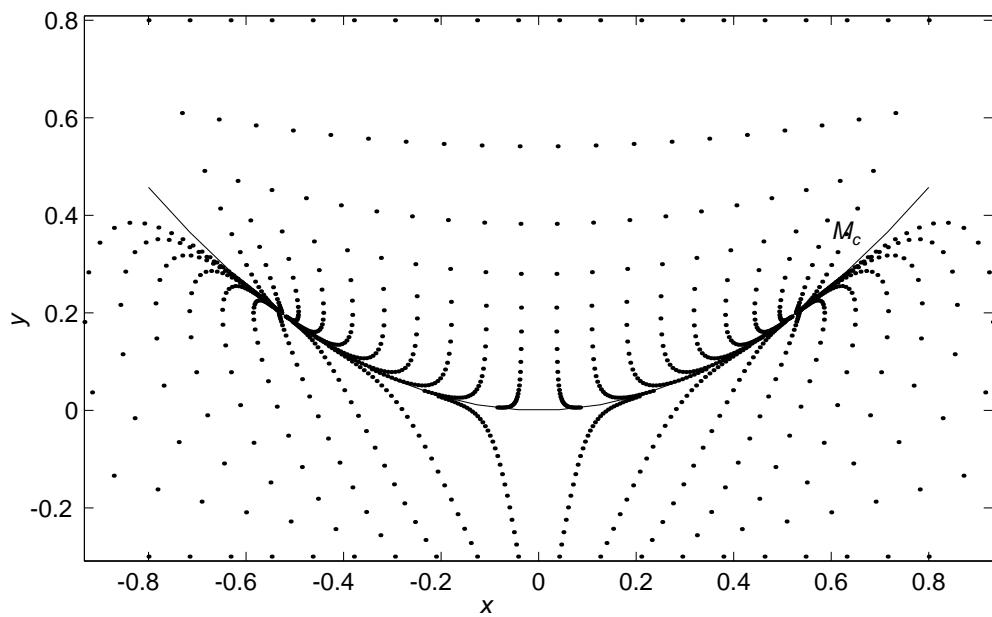


Figure 2: trajectories of the dynamical system (2) with parameter $\epsilon = 0.2$ plotted every $\Delta t = 0.2$ to show the rapid approach, roughly in 2 units of time, to the centre-unstable manifold, \mathcal{M}_{cu} . Again observe the collapse onto a one-dimensional set of states.

et al [3] to investigate the Kuramoto-Sivashinsky dynamics, by Cheng & Chang [36] for subharmonic instabilities of waves, and by Chow & Lu [40] to compare with the method of averaging. For general ϵ , not too large in magnitude, (3) describes an *invariant manifold* [180, §1.1C], which when exponentially attractive may be used to create accurate models [140, 141], such as that of dispersion in simple channels [178]. Invariant manifolds, based upon modes with $\Re(\lambda) > -\gamma$, may improve the numerical solution of spatio-temporal PDES through what others have called the *nonlinear Galerkin method* [105, 170, 64, 104, 65]. The disadvantage of these more general invariant manifolds for modelling is that the consequent algebraic analysis in constructing the manifold is significantly more difficult. In §§2.3 I discuss a method to approximate these more general invariant manifold models while maintaining the relatively simple algebra associated with centre manifolds.

A quite different concept in modelling dynamics is what van Kampen [175] calls that of the *guiding centre*. In the presence of fast oscillations, such as short-period waves, we may be only interested in the long-term evolution of the dynamics of the mean. For example, incompressible fluid dynamics ignores fast sound waves (for example, see the quasi-incompressible approximation of Durran [57]). In such a case the particular dynamical flow without oscillations acts as a centre for flow with rapid oscillations and so is considered to form a low-dimensional model. In many applications, such as beam theory [145] or atmospheric flows [102, e.g.], the model dynamics are known as the slow dynamics on the *slow manifold*. However, the modelling issues associated with this concept are much more delicate and I discuss them in more detail in §5. For the moment let us concentrate upon exponential collapse to the centre manifold as a rationale for forming low-dimensional dynamical models.

A completely different approach, albeit justified by the same exponential collapse, is to use symmetry considerations to sketch out possible structurally stable, low-dimensional models. See [60] for example, or the review by Crawford & Knobloch [50] on symmetry in fluid dynamics. The limitation of this approach is that one can practically never determine quantitative coefficients in the model. Hence such an approach may be satisfactory for predicting the broad range of phenomena possible, but it cannot produce a model able to make detailed forecasts about a specific physical situation.

2.2 Centre manifolds

The centre manifold, \mathcal{M}_c , is the curved “subspace” to which all trajectories in the neighbourhood of a fixed point are attracted exponentially quickly. Being composed of trajectories of the TOE and being low-dimensional, the

evolution on \mathcal{M}_c qualifies as forming a model of the TOE.

Three theorems claim:

1. \mathcal{M}_c exists for the previously mentioned structure of eigenvalues, provided the nonlinear terms are not “badly” behaved;
2. \mathcal{M}_c is relevant as all solutions in the neighbourhood are attracted exponentially to a solution on \mathcal{M}_c ;
3. \mathcal{M}_c may be constructed to any desired degree of accuracy (asymptotically speaking).

Detailed proofs of these theorems have, for example, been given in the excellent little book by Carr [26], and more recently by Vanderbauwhede and Iooss [177, 176]. In this subsection I address their role in low-dimensional modelling.

2.2.1 Existence

Most statements of theory address dynamical systems in the *separated form*:

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}), \quad (4)$$

$$\dot{\mathbf{y}} = B\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}), \quad (5)$$

where the eigenvalues of the matrix A have real-part zero, the eigenvalues of the matrix B have strictly negative real-part, and \mathbf{f} and \mathbf{g} are quadratically nonlinear functions at the origin, of $\mathcal{O}(x^2 + y^2)$ where $x = |\mathbf{x}|$ and $y = |\mathbf{y}|$. However, we address dynamical systems, the TOE, in the general form

$$\dot{\mathbf{u}} = \mathcal{L}\mathbf{u} + \mathbf{f}(\mathbf{u}), \quad (6)$$

where $\mathbf{u}(t) \in \mathbb{R}^n$, the linear operator \mathcal{L} has m eigenvalues with zero real-part (counted according to their multiplicity), the remaining eigenvalues are strictly negative and bounded above by $-\gamma < 0$, and \mathbf{f} is a quadratically nonlinear at the origin, of $\mathcal{O}(u^2)$. A linear transformation of variables will in theory transform the dynamical system from one form to the other. However, in practise we prefer to perform analysis with meaningful physical variables and so the general form (6) is more appropriate in applications.

Linearly, according to $\dot{\mathbf{u}} = \mathcal{L}\mathbf{u}$, all the dynamics will collapse exponentially quickly onto the *centre subspace*,

$$\mathcal{E}_c = \text{span} \{ \mathbf{e}_1, \dots, \mathbf{e}_m \},$$

where \mathbf{e}_j are the m eigenvectors (or generalised eigenvectors) of the m eigenvalues with zero real-part (counted according to multiplicity). Theory asserts

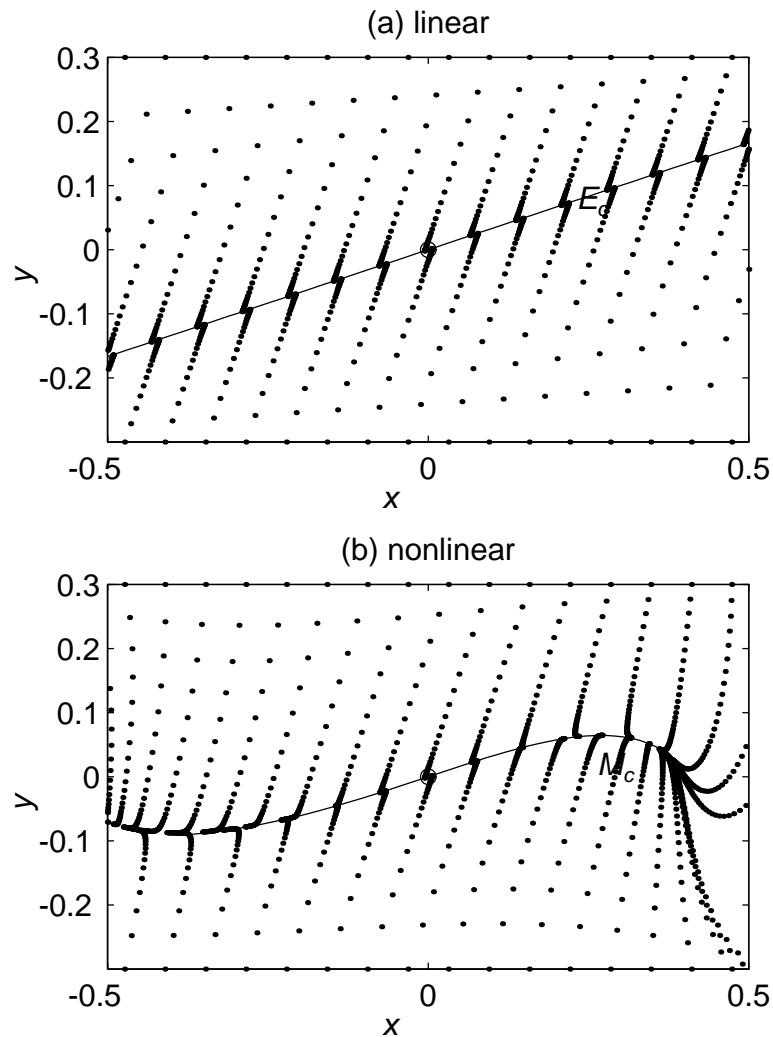


Figure 3: shows an example of the exponential collapse of (a) linear dynamics onto the centre subspace \mathcal{E}_c , each dot on a trajectory a fixed time Δt apart, to compare with (b) showing that the nonlinear dynamics “bend” the subspace to the centre manifold \mathcal{M}_c and produce a slow evolution thereon.

that at least near the origin the nonlinear terms just “bend” this invariant subspace and modify the dynamics on the subspace, as shown in Figure 3.

Theorem 1 (existence) *sufficiently near the origin, in some neighbourhood U , there exists an m -dimensional invariant manifold for (6), \mathcal{M}_c , with tangent space \mathcal{E}_c at the origin, in the form $\mathbf{u} = \mathbf{v}(\mathbf{s})$ (that is, locations on \mathcal{M}_c are parameterised by a set of variables \mathbf{s} , often called the order parameters). The flow on \mathcal{M}_c is governed by the m -dimensional dynamical system*

$$\dot{\mathbf{s}} = \mathcal{G}\mathbf{s} + \mathbf{g}(\mathbf{s}), \quad (7)$$

where \mathcal{G} is the restriction of \mathcal{L} to \mathcal{E}_c , and the nonlinear function \mathbf{g} is determined from \mathcal{M}_c and (6). Because of the nature of the eigenvalues of \mathcal{L} this invariant manifold is called a centre manifold of the system.

A centre manifold is, in some neighbourhood of the origin, at least as differentiable as the nonlinear terms \mathbf{f} . However, it may not be analytic even though \mathbf{f} is analytic. Also, a centre manifold need not be unique, but the differences between the possible centre manifolds are of the same order as the differences we set out to ignore in establishing the low-dimensional model [137, 140]. Non-uniqueness, when it arises, is irrelevant for modelling.

In application, such as the fluid dynamics problems discussed in §3, we need theory dealing with not only infinite dimensional TOE’s, but also infinite dimensional centre manifolds. Extensions of the above theorem to infinite dimensional dynamics are extant but limited. The most general theory I am aware of is currently due to Gallay [71], but it suffers from restrictions upon the nonlinear terms, they have to be bounded, which limits its rigorous application. Scarpellini [151] apparently places significantly less restrictions upon the nonlinearities in the dynamical equations, but while he addresses infinite dimensional centre manifolds, the results are severely constrained by requiring finite dimensional stable dynamics. Härägus [85, 86] has developed theory supporting infinite dimensional models, such as the Korteweg-de Vries equation, but only by placing extremely limiting restrictions upon the time derivatives in the TOE.

2.2.2 Relevance

Based on the rationale of neglecting rapidly decaying transients, our intention is to consider (7) as a simple model system for the TOE (6); simpler in the sense that it has lower dimensionality, m instead of n . However, there is a subtle point which is often overlooked. We must be assured that the actual solutions of the model (7) do indeed correspond to solutions of the

full system (6). Exponential attraction to an invariant manifold is *not* enough to assure us that the solutions on the manifold actually model accurately all the nearby dynamics.¹

Theorem 2 (relevance) *The neighbourhood U may be chosen so that all solutions of (6) staying in U tend exponentially to some solution of (7). That is, for all solutions $\mathbf{u}(t) \in U$ for all $t \geq 0$ there exists a solution $\mathbf{s}(t)$ of the model (7) such that*

$$\mathbf{u}(t) = \mathbf{v}(\mathbf{s}(t)) + \mathcal{O}(e^{-\gamma't}) \quad \text{as } t \rightarrow \infty,$$

where $-\gamma'$ may be estimated as $-\gamma$, the upper bound on the negative eigenvalues of the linear operator \mathcal{L} .

This theorem is crucial to modelling; it asserts that for a wide variety of initial conditions the dynamics of the TOE decays exponentially quickly to a solution which can be predicted by the low-dimensional model. Somewhat pessimistically, the theorem requires initial conditions to be sufficiently small, namely in U . But consider the system shown in Figure 1. Observe that all trajectories within the picture asymptote exponentially to the centre manifold. Thus the conclusions of this theorem are correct for at least *all* initial conditions within the figure. In practice the neighbourhood U may be quite large.

2.2.3 Approximation

We need to find an equation to solve which gives the centre manifold \mathcal{M}_c . This is obtained straightforwardly by substituting the assumed functional relations, that $\mathbf{u}(t) = \mathbf{v}(\mathbf{s}(t))$ where $\dot{\mathbf{s}} = \mathcal{G}\mathbf{s} + \mathbf{g}(\mathbf{s})$, into the TOE (6) and using the chain rule for time derivatives to obtain

$$\mathcal{L}\mathbf{v}(\mathbf{s}) + \mathbf{f}(\mathbf{v}(\mathbf{s})) = \frac{\partial \mathbf{s}}{\partial \mathbf{v}} [\mathcal{G}\mathbf{s} + \mathbf{g}(\mathbf{s})]. \quad (8)$$

This is the equation to be solved for the centre manifold \mathcal{M}_c .

An extra condition is that \mathcal{E}_c is the tangent space of \mathcal{M}_c at the origin. Put more crudely, this requires that \mathbf{v} is quadratically close to \mathcal{E}_c . This condition ensures that the constructed manifold truly contains the whole of

¹For example, consider the system $\dot{x} = x + xy$ and $\dot{y} = -y$ which has the exponentially attractive unstable manifold $y = 0$. There is generally an unavoidable $\mathcal{O}(1)$ discrepancy between the full system, with solutions $x(t) \sim Ae^t + B$, and the low-dimensional model, $\dot{x} = x$ with $x(t) \sim Ae^t$, despite the exponential collapse onto the unstable manifold.

the critical centre modes, and nothing but the centre modes. Without it, the solution of (8) could be based on an almost arbitrary mixture of linear modes. Indeed, other invariant manifolds of note satisfy (8) but are tangent to different subspaces as $\mathbf{s} \rightarrow 0$.

It is typically impossible to find exact solutions to (8). However, in applications we may approximate \mathcal{M}_c to any desired accuracy. For functions $\phi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ (imagine that ϕ approximates the shape \mathbf{v}) and $\psi : \mathbb{R}^m \rightarrow \mathbb{R}^m$ (imagine that ψ approximates the evolution $\dot{\mathbf{s}} = \mathcal{G}\mathbf{s} + \mathbf{g}(\mathbf{s})$) define the *residual* of (8)

$$R(\phi, \psi) = \frac{\partial \phi}{\partial \mathbf{s}} \psi(\mathbf{s}) - \mathcal{L}\phi(\mathbf{s}) - \mathbf{f}(\phi(\mathbf{s})), \quad (9)$$

and observe that \mathcal{M}_c satisfies $R(\mathbf{v}(\mathbf{s}), \mathcal{G}\mathbf{s} + \mathbf{g}(\mathbf{s})) = 0$.

Theorem 3 (approximation) *If the tangent space of $\phi(\mathbf{s})$ at the origin is \mathcal{E}_c , and the residual $R(\phi, \psi) = \mathcal{O}(s^p)$ as $s \rightarrow 0$ for some $p > 1$ (where $s = |\mathbf{s}|$) then $\mathbf{v}(\mathbf{s}) = \phi(\mathbf{s}) + \mathcal{O}(s^p)$ and $\dot{\mathbf{s}} = \mathcal{G}\mathbf{s} + \mathbf{g}(\mathbf{s}) = \psi(\mathbf{s}) + \mathcal{O}(s^p)$ as $s \rightarrow 0$.*

That is, if we can satisfy (8) to some order of accuracy then the centre manifold is given to the same order of accuracy.

In problems specified in the separated form (4–5) the centre manifold may be determined simply by iteration [26, e.g.]. However, typically a solution is sought in the form of an asymptotic power series in \mathbf{s} as developed by Coullet & Spiegel [44] for the general form (6) (and reinvented by Leen [98]). Such a power series solution looks very like the Lyapunov-Schmidt method [83, e.g.] for determining the nontrivial fixed points near a simple bifurcation; in contrast though, centre manifold theory also determines the dynamics, see [9, 167, 119, 39] for comparisons. Recently, I developed [148] an iterative algorithm, for the general form, with the virtue that it is readily implemented in computer algebra, see §9.

Centre manifold models need not be anchored to just the one equilibrium. Some very interesting models are constructed for the dynamics in the neighbourhood of a manifold of equilibria. Such models may be uniformly valid across the whole set of equilibria. One example is given in §§3.3 where a fluid at rest of constant but *arbitrary* thickness provides the reference equilibria for a model of the evolution of the fluid film's thickness.

2.3 A vital extension

Requiring that $\Re(\lambda) = 0$ precisely as the criterion for the modes of the dynamical model is far too restrictive in practice. A trick rescues the theory from obscurity.

It is easiest to see the trick in an example: to (2) adjoin the trivial dynamical equation $\dot{\epsilon} = 0$. Based on the linearisation about the equilibrium at the origin in the ϵxy -space (note that in this space ϵx is a nonlinear term), there then exists a two-dimensional centre manifold, parameterised by ϵ and x . The relevance and approximation theorems may similarly be applied to construct the centre manifold for small x and ϵ , and to validate the model, that

$$\dot{\epsilon} = 0, \quad \text{and} \quad \dot{x} = \epsilon x - x^3 + \mathcal{O}(s^4),$$

where $s = |(\epsilon, x)|$. Fixing upon any small value for ϵ then gives a model involving only the evolution of x , from which one may predict, for example, the presence of a pitchfork bifurcation.

This example shows that we may apply centre manifold theory to problems which do not precisely fit the linear structure required by the theory. By adjoining trivial dynamical equations we may, in essence, perturb eigenvalues with small real-part, either negative or positive, so that the “interesting” modes then have eigenvalues that precisely satisfy $\Re(\lambda) = 0$ and hence are included in the centre manifold. This trick can be not only used to unfold bifurcations, as above, it may also be used to partially justify the long-wave approximation (§§3.2), and to approximate “hard” problems by writing them as perturbations of easier problems (§§3.3).

Because parameters are so important in application, it is useful to quote a more powerful and flexible approximation theorem. Consider the extended dynamical system with ℓ parameters ϵ :

$$\dot{\epsilon} = \mathbf{0}, \quad \text{and} \quad \dot{\mathbf{u}} = \mathcal{L}\mathbf{u} + \mathbf{f}(\epsilon, \mathbf{u}), \quad (10)$$

with \mathcal{L} as before. For functions $\phi : \mathbb{R}^\ell \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ approximating the centre manifold, and $\psi : \mathbb{R}^\ell \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ approximating the evolution thereon, one may argue that

Theorem 4 (parameter approximation) *If the tangent space of $\phi(\epsilon, \mathbf{s})$ at the origin is $\mathbb{R}^\ell \times \mathcal{E}_c$, and the residual of (10) is $\mathcal{R}(\phi, \psi) = \mathcal{O}(s^p + \epsilon^q)$ as $(\epsilon, \mathbf{s}) \rightarrow \mathbf{0}$ for some $p > 1$ and $q \geq 1$, then $\mathbf{v}(\epsilon, \mathbf{s}) = \phi(\epsilon, \mathbf{s}) + \mathcal{O}(s^p + \epsilon^q)$ and $\dot{\mathbf{s}} = \mathcal{G}\mathbf{s} + \mathbf{g}(\epsilon, \mathbf{s}) = \psi(\epsilon, \mathbf{s}) + \mathcal{O}(s^p + \epsilon^q)$ as $(\epsilon, \mathbf{s}) \rightarrow \mathbf{0}$. Arguments also justify the same statement but with errors $\mathcal{O}(s^p, \epsilon^q)$.*

Consequences of this property are followed up in §4 where I discuss the flexibility of centre manifold models.

3 Physical applications show the potential

Centre manifold analysis has often been applied to reduce dynamics down to a “handful” of ODEs, typically the centre manifold is of dimension 1–3. Also typically, a normal form transformation is then used to classify the dynamics obtained on the centre manifold. This established approach is described in various books dealing with dynamical systems, see Carr [26], Guckenheimer & Holmes [79, Ch.3], Wiggins [180, Ch.2], Iooss & Adelmeyer [88, Ch.1], or Kuznetsov [96, Ch.3–5]. However, there is much greater scope for the use of centre manifold analysis in low-dimensional modelling. After a brief dash through some standard applications of the theory, in the rest of this review I outline some of the interesting applications that extend beyond current rigorous theory.

3.1 A dash of straightforward applications

- Fluid mechanics is a haven of nonlinear dynamics and the application of centre manifold theory. The theory has been used in analysing the dynamics of Taylor vortices in Taylor-Couette flow [97, 88, 88], and the non-axisymmetric dynamics [38, 84] involving mode competition. Mode interactions in rotating convection [116] are analysed with centre manifolds, as are the dynamics of convection in porous media by Neel [123, 124, 125] and others [74]. Arneodo *et al* [6, 5, 4, 9] reduced the dynamics of triple convection down to a set of three coupled ODEs, numerically verified the modelling and then proved the existence of chaos.
- A number of studies [13, 34, 18, 35, e.g.] have used centre manifold theory to help analyse the effectiveness of feedback control in nonlinear dynamics [92], sometimes the centre manifold is called the slow manifold [93, 58, 161], see §5. These control problems are often concerned with systems where the control is delayed, leading to formally infinite-dimensional dynamics, and they investigate the effect of the delay on the dynamics through a low-dimensional centre manifold model [15, 16, 25, e.g.]. Some studies examine the centre manifolds of delay differential equations in their own right, for example: existence in PDEs [100], and the effects of multiple delays [82, 14]. Usually, in these applications any finite time delay is washed out by the slow-evolution paradigm on the centre manifold. However, in some other applications (see §§3.2) it is attractive to transform a centre manifold model into an equivalent model which possesses delays or “memory.”

- Other mechanical systems of interest involve nonlinear elastic [77, 35] and viscoelastic [165, 164] springs. The usual aim is to determine the onset and form of nonlinear oscillations, such as airfoil flutter [77] or chaos in pipes conveying fluid [99], near some reference equilibrium.
- The field of economics is increasingly using the concepts and techniques of nonlinear dynamics, as described by Chiarella [37], and centre manifold theory has a role to play [12, e.g.].
- A remarkable application occurs in determining *steady* solutions in large physical domains. As developed by Mielke to explore Saint-Venant’s principle in elasticity [113, 111, 112], the spatial dimension is treated as a “time” variable and centre manifold theory applied. The analysis elucidates spatial structures in the problem, such as the Ginzburg-Landau equation for nearly periodic solutions. Applications of this idea have been to: internal waves of a stratified fluid in a two-dimensional channel and capillary surface waves [114]; Poiseuille flow between parallel plates [1]; hydrodynamic stability in an infinitely long cylinder by Iooss *et al* [90, 89]; the dynamics of Taylor-Couette flow also by Iooss & Adelmeyer [88]; and bifurcations and other behaviour near spatially periodic structures rather than spatially constant [91, 115]. This trick is also employed, see §8, to determine boundary conditions for models of spatio-temporal dynamics.

In the last of the above mentioned applications there is no time dependence, no dynamics! This is a disappointing limitation. However, by stretching the centre manifold techniques and concepts past the rigorous extent of current theory, we may form models of spatio-temporal dynamics. In problems with large-spatial extent, as discussed in the next three subsections, the resulting centre manifold will be of “infinite dimension”. Nonetheless, the model will be significantly simpler in one or more characteristics than the “infinite dimensional” TOE from which it is derived—such models are useful.

3.2 Slow space variations—dispersion in a channel

In a long, thin channel or pipe, as shown schematically in Figure 4, the dominant process of dispersion of a contaminant is diffusion across the thin channel. This leads to decay of all modes except the constant mode (across the channel). But at different stations the constant will typically be different, and so the cross-sectional mean concentration C will depend upon downstream distance x . The shearing then causes these variations in x to

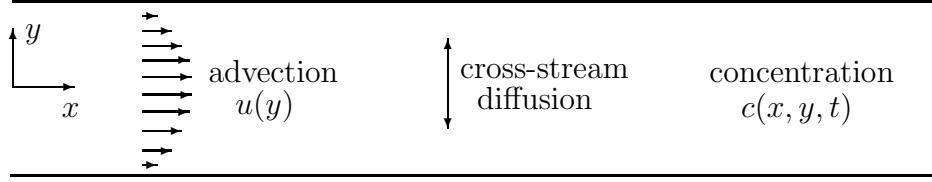


Figure 4: schematic diagram of the dispersion of a contaminant with concentration field $c(x, y, t)$ in a channel with advection $u(y)$ and cross-stream diffusion.

move and interact leading to the Taylor model [168, 169]

$$\frac{\partial C}{\partial t} = -U \frac{\partial C}{\partial x} + D \frac{\partial^2 C}{\partial x^2} + \dots, \quad (11)$$

where U is simply the mean downstream velocity, but the effective diffusivity D is a nontrivial function of the velocity field. The Taylor model predicts the long-term concentration is in the shape of a moving and spreading Gaussian. This model is derived rigorously via centre manifold theory using an interesting trick introduced in [138].

The simplest example is the physical problem shown in Figure 4 which has the non-dimensional governing equation

$$\frac{\partial c}{\partial t} = -u(y) \frac{\partial c}{\partial x} + \delta \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}, \quad (12)$$

where, for definiteness, the advecting velocity may be $u(y) = 3(1 - y^2)/2$ in a channel $-1 < y < 1$, and where δ , the inverse square of a Peclet number, is typically very small. Now take the Fourier transform of (12), so that $c = \int_{-\infty}^{\infty} \hat{c} e^{ikx} dx$, to obtain the Fourier space equation

$$\frac{\partial \hat{c}}{\partial t} = -iku(y)\hat{c} - \delta k^2 \hat{c} + \frac{\partial^2 \hat{c}}{\partial y^2}. \quad (13)$$

Then utilise the trick introduced in §2.3 to unfold bifurcations, but here to introduce the approximation of a *large-scale, slowly-varying* spatial dependence. The trick is to adjoin to (13) the trivial equation that the wavenumber k is constant:

$$\frac{\partial k}{\partial t} = 0. \quad (14)$$

Adjoining this trivial equation focuses the centre manifold analysis upon the small wavenumber, large-scale dynamics in the channel. The advection term

$-iku(y)\hat{c}$ is then “nonlinear” in k and \hat{c} . Linearly (in this new sense) cross-channel diffusion causes concentrations \hat{c} to decay exponentially quickly to a constant with respect to y : $\hat{c} \approx \hat{C}$. Centre manifold theory applied to (13–14) then derives, as a matter of course, that the Fourier transform of the cross-channel average, \hat{C} , will in the long-term evolve according to

$$\frac{\partial \hat{C}}{\partial t} = -ik\hat{C} - \left(\delta + \frac{2}{105}\right) k^2 \hat{C} + \mathcal{O}(k^3). \quad (15)$$

Taking the inverse Fourier transform leads to the Taylor model (11), with here the particular coefficients $U = 1$ and $D = \delta + 2/105$.

Observe that the Taylor model (11), while certainly simpler than the TOE (12), is still of “infinite” dimension—there is an infinite freedom in $C(x)$ that is governed by the model. In such “infinite” dimensional centre manifolds there is a much richer field of possible and applicable models to investigate than in centre manifolds of just a few dimensions.

It is straightforward for the centre manifold analysis to compute higher order models of the dispersion, involving terms such as C_{xxx} and C_{xxxx} for example, that show the evolution of the skewness and kurtosis of the concentration distribution. Indeed, Mercer & I have extended the analysis of the dispersion in a channel [109] and pipe [110] to very high order and shown, using a generalisation of the Domb-Sykes plot [109, Appendix], that these models actually converge for non-dimensional downstream wavenumber $|k|$ less than roughly 10, depending upon the particular problem in hand (similar bounds also have been computed for beam theory, see §5 and [145]). Thus here we are assured that the model for dispersion resolves spatial details down to roughly $2\pi/10$ times the downstream advection distance in a cross-stream diffusion time. This quantitative limit on the resolution is some 10 times better than that estimated by Taylor. The Relevance Theorem also assures us that the model will resolve temporal dynamics longer than a cross-stream diffusion time, the time-scale of approach to the centre manifold. Both of these limits to the resolution of the model can only be improved by retaining more dynamic modes in the model, as was done by Watt & I [178] in investigating an invariant manifold model based upon the two or three gravest modes in the channel. That we can sometimes find these *quantitative* bounds to the domain of the validity of a low-dimensional model is part of the power of centre manifold theory.

The above analysis in Fourier space is more-or-less rigorous. However, upon inspecting the details of the algebra it is apparent that the wavenumber, in the combination ik , just acts as a place holder for the spatial derivative $\partial/\partial x$. The adjoining of the trivial dynamical equation (14), that $\partial k/\partial t = 0$, just focusses attention upon small wavenumber. Precisely the same effect is

achieved in physical variables simply by treating $\partial_x = \partial/\partial x$ as “small.” A rationale for doing this is outlined in [138] based upon the local dynamics in any reach of the domain, but rigorous it is not because the operator ∂_x is unbounded and the extant theory [71] cannot be applied directly. However, an analysis based upon this idea has the advantage that one deals in physical variables, rather than Fourier transformed quantities, and it easily generalises to more difficult problems as seen in the next two subsections. For example, one may analyse [109, 110] the effect upon the Taylor model of spatial and temporal variations in the flow-rate, width of the channel, and cross-stream diffusion. Interestingly, as first pointed out by Smith [162], the results may be recast in terms of a memory of the conditions upstream or at earlier times.

The centre manifold analysis of dispersion in a pipe has also been extended by Balakotaiah & Chang [10] to the dynamics in a chromatograph chemical reactor where chemical reactions occur either in the flow or with the walls of the pipe.

3.3 Cross-sectional averaging is unsound—thin film flows

The dynamics of thin films of fluid are important in many industrial, environmental and biological processes. An approximation of such a thin viscous fluid flow, as shown schematically in Figure 5, with slow spatial variations leads to a Kuramoto-Sivashinsky type of equation. If surface tension σ is the only driving force then the simplest long-wave model is

$$\frac{\partial \eta}{\partial t} \approx -\frac{\sigma}{3\mu} \frac{\partial}{\partial x} \left(\eta^3 \frac{\partial^3 \eta}{\partial x^3} \right), \quad (16)$$

where μ is the fluid viscosity and $\eta(x, t)$ is the film thickness above a flat substrate.

To analyse the Navier-Stokes equations, the TOE, to derive such a model [149], recognise that across the thin fluid film, viscosity acts quickly to damp almost all cross-film structure. The distinction between the longitudinal and cross fluid dynamics is exactly analogous to that of dispersion in a pipe; we proceed with a similar analysis. However, the equations are very different (see Figure 5). In particular, this problem has many nonlinearities: not only is the advection in the Navier-Stokes equation described by a nonlinear term, but also the thickness of the fluid film is to be found as part of the solution and its unknown location is another source of nonlinearity. Here, the only practical course of analysis is to deal with physical variables and treat ∂_x as a “small” parameter which is negligible to the leading “linear” approximation (though recall that Härägus [86] otherwise justifies the Korteweg-de Vries equation for the *inviscid*, long-wave hydrodynamics).

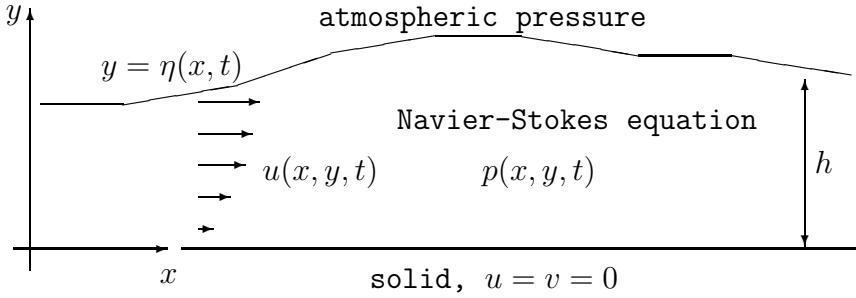


Figure 5: schematic diagram of a thin fluid film flowing down a solid bed.

Assuming no longitudinal variations at all, a linear analysis of the equations shows that there is one critical mode in the cross-fluid dynamics, all others decay exponentially due to viscosity. This critical mode is associated with conservation of the fluid and consequently it is natural to express the low-dimensional model in terms of the film thickness $\eta(x, t)$. An interesting aspect is the fact that although this is a nonlinear problem, conservation of fluid applies no matter how thick the fluid layer, and the state of no flow is an equilibrium for all constant η . Thus the analysis is valid for arbitrarily large variations in the thickness of the film, just so long as the variations are sufficiently slow in space and time. A relatively simple toy example of this is also discussed in [138]. These are examples, as mentioned in §§§2.2.3, of a centre manifold analysis based upon a manifold of equilibria rather than being anchored to just one fixed point.

Treating ∂_x as a small perturbing operator, the centre manifold formalism follows to straightforwardly derive models such as (16). Of course, higher order models may be straightforwardly constructed if necessary [148], viz

$$\begin{aligned} \frac{\partial \eta}{\partial t} = & -\frac{\sigma}{\mu} \frac{\partial}{\partial x} \left[\frac{1}{3} \eta^3 \eta_{xxx} + \frac{3}{5} \eta^5 \eta_{xxxxx} + 3\eta^4 \eta_x \eta_{xxxx} \right. \\ & \left. + \eta^4 \eta_{xx} \eta_{xxx} + \frac{11}{6} \eta^3 \eta_x^2 \eta_{xxx} - \eta^3 \eta_x \eta_{xx}^2 \right] + \mathcal{O}(\partial_x^7). \end{aligned}$$

However, much more interesting is to model the 2D spread of a 3D fluid over a curved substrate [149]. Provided that, as in dispersion in a varying width pipe or channel [109, 110], the length-scale of curvature variations is large compared to the film thickness, a similar analysis leads to the model

$$\frac{\partial \zeta}{\partial t} \approx -\frac{\sigma}{3\mu} \nabla \cdot \left[\eta^2 \zeta \nabla \tilde{\kappa} - \frac{1}{2} \eta^4 (\kappa \mathbf{I} - \mathbf{K}) \cdot \nabla \kappa \right], \quad (17)$$

where: $\zeta = \eta - \frac{1}{2}\kappa\eta^2 + \frac{1}{3}k_1k_2\eta^3$ is proportional to the amount of fluid locally “above” the substrate; $\tilde{\kappa}$ is the approximate mean curvature of the free-surface; \mathbf{K} is the curvature tensor of the substrate; k_1 , k_2 , and $\kappa = k_1 + k_2$ are the principal curvatures and the mean curvature of the substrate, respectively; and the ∇ -operator is expressed in a coordinate system of the curving substrate. The effects of gravity and fluid inertia also may be included systematically [149, §4] to produce a more extensive model.

One significant limitation in the use of the whole family of models discussed above is that they only resolve dynamics significantly slower than the time-scale of viscous decay, $T \approx 0.4\eta^2/\nu$, of the gravest shear mode, $u_1 \propto \sin(y\pi/2\eta)$. For thicker fluid films, interesting dynamics occurs on about this time-scale. Another class of models is needed in such a situation. Traditionally these are obtained by integrating or averaging the horizontal momentum equation over the depth of the fluid to obtain an evolution equation for the mean horizontal velocity $\bar{u}(x, t)$ [133, 30, e.g.].

Unfortunately, such simple modelling methods do not just get the coefficients wrong, they can fail dramatically. Abstractly, cross-sectional averaging is a projection of the dynamical equations onto some linear subspace of the state space, namely the space of functions which are constant over cross-sections. Typically, linear considerations suggest that this would be a useful procedure. But consider the simple example [96, p153]

$$\dot{x} = xy + x^3, \quad \text{and} \quad \dot{y} = -y - 2x^2, \quad (18)$$

whose trajectories are plotted in Figure 6. Linearly, $\dot{x} = 0$ and $\dot{y} = -y$ and so solutions exponentially quickly approach the subspace $y = 0$. Thus, projecting the dynamics onto $y = 0$ to give $\dot{x} = x^3$ may be expected to give a reasonable model of the long-term dynamics of the system from which one would deduce that $x = 0$ is an unstable fixed point. But this is not so. The centre manifold is $y = -2x^2 - 4x^4 + \mathcal{O}(x^6)$ whence we deduce the correct model of the long-term dynamics to be $\dot{x} = -x^3 + \mathcal{O}(x^4)$ and hence the origin is actually stable, as seen in Figure 6. *Cross-sectionally averaging dynamical equations is unsound as a modelling paradigm.*

Returning to the dynamics of a fluid film, observe that in the horizontal shear modes, $u_n \propto \sin(ny\pi/2\eta)$ there is a relatively large *spectral gap* between the gravest mode u_1 and the next mode u_2 ; the eigenvalues are $\lambda_1 = -2.5\nu/\eta^2$ and $\lambda_2 = -22.2\nu/\eta^2$ respectively. Surely we should be able to gather the u_1 mode into the centre manifold.

The trick of §§2.3 allows us to apply centre manifold techniques to obtain a coupled model for the evolution of \bar{u} and η which resolves transients on a shorter time-scale. Manipulating the horizontal momentum equation [147]

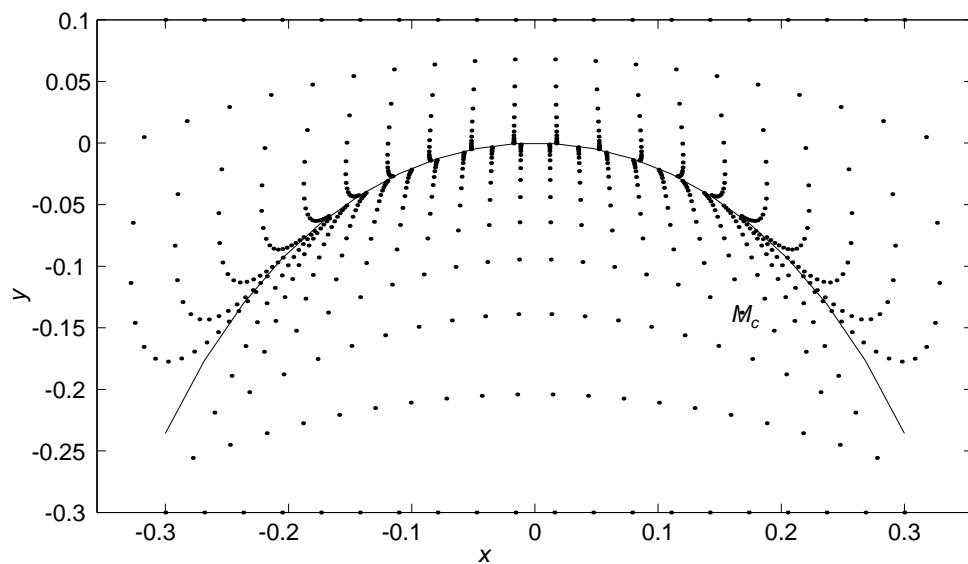


Figure 6: trajectories of the dynamical system (18), plotted as dots $\Delta t = 0.4$ apart, showing that although trajectories point away from the origin on $y = 0$, nonetheless the nonlinear dynamics on the centre manifold, \mathcal{M}_c , shows that the origin is stable.

to

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \frac{\partial p}{\partial x} + \nu \nabla^2 u + (1 - \gamma) \nu \left(\frac{\pi}{2\eta} \right)^2 u, \quad (19)$$

and adjoining $\dot{\gamma} = 0$ changes the spectrum. When $\gamma = 0$, the introduced artificial forcing makes u_1 a critical mode, along with η , and hence there exists a centre manifold parameterised by \bar{u} , measuring the amplitude of u_1 , the film thickness η , and the artificial parameter γ . Setting $\gamma = 1$ recovers the original PDES and so pursuing the analysis and subsequently setting $\gamma = 1$ leads to an approximate model for the original dynamics. The model is found [147] to be

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}(\bar{u}\eta), \quad (20)$$

$$\frac{\partial \bar{u}}{\partial t} \sim 0.8238(-g\eta_x + \sigma\eta_{xxx}) - 1.504\bar{u}\bar{u}_x - 2.467\frac{\nu}{\eta^2}\bar{u} - 0.1516\frac{\bar{u}^2}{\eta}\eta_x \quad (21)$$

as in depth averaged equations but quantitatively different. The adaptation (19) of the horizontal momentum equation is not unique. Exactly the same eventual model (21) is obtained [146] by appropriate manipulation of the tangential stress boundary condition at the free-surface.

Similar ideas are being employed to analyse the dynamics of a turbulent river or flood. Initial work by Mei & I is reported in [107] where we took the k - ϵ model of turbulent flow as the TOE, and perturbed some critical coefficients so that a centre manifold inspired model could be constructed based upon the water depth η and the depth averages of the horizontal velocity \bar{u} , the turbulent energy \bar{k} , and the turbulent dissipation $\bar{\epsilon}$. Further analysis on this interesting model is in progress; it promises a sophisticated and reliable model of flood, river and estuarine dynamics.

However, we need more powerful theory on infinite dimensional centre manifolds to provide rigorous support for these sorts of interesting slowly-varying, long-wave models. I tell my graduate students that as far as rigorous theory is concerned: for simple bifurcations we are on solid ground, for shear dispersion we may well be on thin ice, but in the application to thin film flows we are walking on water!

3.4 Bands of critical modes—convection

In a fluid layer heated from below, diffusion may damp all motion. If the heating is large enough then warm light fluid rises and cool heavy fluid falls. If the top and bottom boundaries are insulating, the case of fixed heat flux, as elaborated by Proctor *et al* [33, 32, 132, 56, 136], the convection occurs

on a large horizontal length-scale, everywhere small wavenumber, and so it may be modelled by the approach of the previous subsection.

But in the usual case of fixed temperature top and bottom boundary conditions, the fluid flow occurs on a horizontal length-scale of the same size as the height of the fluid layer. Near the critical temperature difference (measured by the Rayleigh number R), and using the trick of §§2.3 again, a centre manifold may be found consisting primarily of a superposition of “rolls”, say

$$u \approx A\ell e^{ikx} \cos(\ell z) + \text{c.c.}, \quad \text{and} \quad v \approx Aik e^{ikx} \sin(\ell z) + \text{c.c.}, \quad (22)$$

for some horizontal wavenumber k and some vertical wavenumber ℓ , where A is the complex amplitude of the rolls and “c.c.” denotes the complex conjugate of the appearing terms. The centre manifold analysis [2, 116, e.g.] produces a *Landau equation* for the amplitude such as

$$\frac{dA}{dt} = (R - R_c) A - \alpha |A|^2 A + \mathcal{O}(A^4 + \epsilon^2). \quad (23)$$

However, there are serious problems with such an application in the usual interesting case of convection with large horizontal extent. These problems are endemic to a wide range of pattern evolution problems as discussed in the review by Newell *et al* [126]. The critical spectrum typically looks like that plotted in Figure 7. Almost all modes decay rapidly and so we should be able to form a low-dimensional model. However, there are a continuum of modes with wavenumbers k close to the critical (here $k_c = 1/\sqrt{2}$), and for $R > R_c$ a finite band of these modes become weakly unstable. As in the previous subsections, there should be some way to use the centre manifold techniques and concepts to justify creating a low-dimensional model such as (23).

One avenue of rigorous application of centre manifold theory is mentioned in the start of this section. If we choose to seek just *steady* convective patterns, then we may treat the spatial variable x as a “time-like” variable and construct [91, 115] ODEs that model the spatial “oscillations” of the roll type structure in convection. Unfortunately, although useful for some purposes this approach seems rather limited: it does not address the dynamics, just the equilibria; and there seems no straightforward generalisation to the analysis of the 2D planform adopted by a 3D fluid.

At this stage, one useful approach to model the dynamics is to involve a mixture of centre manifold techniques and ideas from the method of multiple scales. There is no real rigor as yet in general. Use two space scales: the short space scale x to resolve the small-scale structure of the rolls; and a large space scale, X say, to resolve the large scale modulation of the rolls implied

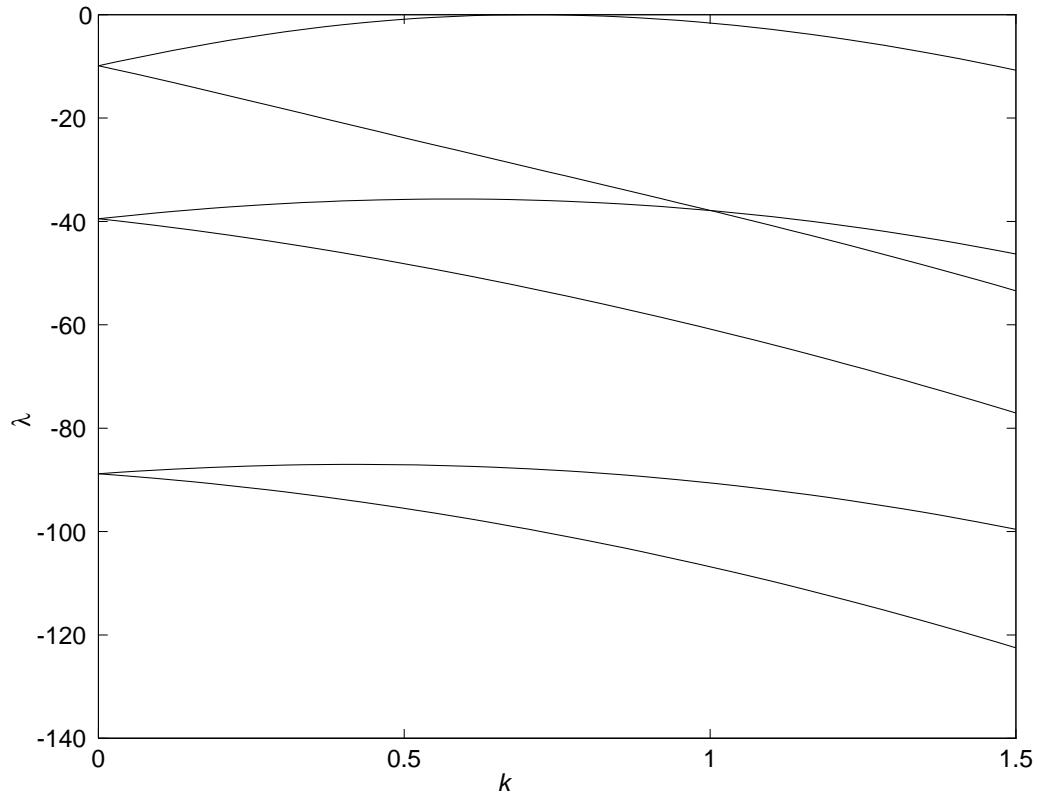


Figure 7: eigenvalues of all modes in convection exactly at critical Rayleigh number versus the continuum of horizontal wavenumbers, k , possible in a layer of large extent. The different branches denote different modes in the vertical. This particular graph is for stress free boundaries and a Prandtl number of 1.

by the band of unstable wavenumbers. Then horizontal derivatives, ∂_x in the Navier-Stokes equations, the TOE, become $\partial_x + \partial_X$. We then treat ∂_X as “small” just as in the previous subsections. The centre manifold algebra then proceeds straightforwardly to deduce a *Ginzburg-Landau* model

$$\frac{\partial A}{\partial t} \approx (R - R_c) A - \alpha |A|^2 A + \beta \frac{\partial^2 A}{\partial X^2} + \dots, \quad (24)$$

for the amplitude of the rolls. Such Ginzburg-Landau models feature prominently in investigations of pattern evolution, see the review by Cross & Hohenberg [53, §III.C.2.d]. Recently Eckhaus [59] has proved that the Ginzburg-Landau equation is indeed relevant to one-dimensional pattern evolution. This centre manifold approach is superior to that of multiple scales alone, not only because of the better geometric viewpoint of centre manifold theory, but also because higher-order corrections may be computed without introducing even further space and time scales [70, Eqn.(2.3),e.g.]—just the one extra space scale is sufficient. Again, however, more theory is needed to support the algebraic formalism in the application of this idea to general problems and their models.

The derivation of models of the spatially 1D evolution of patterns, such as the Ginzburg-Landau model (24) for 2D fluid convection, is reasonable. However, the low-dimensional modelling of patterns in two spatial dimensions, as in 3D fluid convection, is considerably more subtle [126]. To investigate 2D pattern evolution, it is natural to look at the modes and their interactions in 2D Fourier space with wavenumber \mathbf{k} . The difficulty in 2D pattern evolution, and I use convection as a specific example, seems to stem from the fact that the critical modes do not come from a localised band of wavenumbers, but from all the way around an annulus that extends a finite size in wavenumber space, $|\mathbf{k}| \approx k_c$. The nonlinear interactions among such an annulus of modes are vastly richer than those among the small lump of critical modes in a 1D pattern evolution. No really satisfactory modelling procedure has yet been developed, at least not to my taste.

A satisfactory model should only resolve the slow evolution of the near critical modes. Failing that, one possibility is to carry some dynamical “dead wood” in the model by also resolving modes, maybe unphysical modes, which are exponentially decaying. This concept is developed in [143] where it was linked to the geometric idea of *embedding a centre manifold* and the evolution thereon within the dynamics of a higher dimensional dynamical system, but nonetheless of lower dimension than the original TOE. There I show that *adiabatic iteration*, namely the repeated application of adiabatic elimination [81, 175, 173], is an effective algebraic procedure to do this embedding. An open question is: are there other, “better” embedding procedures? In

problems of the convection type, the adiabatic iteration embedding procedure leads to a *generalised Swift-Hohenberg equation* [166],

$$\frac{\partial a}{\partial t} \approx (R - R_c)a - (k_c^2 + \nabla^2)^2 a - a\mathcal{G} \star a^2, \quad (25)$$

where $\mathcal{G} \star$ is some particular radially symmetric convolution. The field $a(\mathbf{x}, t)$ has the same critical annulus of modes, $|\mathbf{k}| \approx k_c$, in which the linear dynamics are that of the TOE. Modes away from the critical annulus are irrelevant as they decay rapidly, they are the “dead wood.” The correct interaction among the critical modes is obtained through the correct determination of the nonlocal nonlinearity in the convolution. Such a model is a remarkably accurate predictor of the pattern evolution [142] in a toy convection problem. Interestingly, people who invoke symmetry arguments to derive the Swift-Hohenberg equation generally fail to acknowledge the possibility of such a nonlocal nonlinearity even though it is indeed permitted [76, p30,e.g.].

Such spatio-temporal modelling of pattern evolution is perhaps a precursor to the modelling of turbulence with its variations across a wide range of spatial and temporal scales. One avenue I would love to find time to explore is to express turbulent fluid flow as a field of interacting wavelets and then embed the dynamics within some economical description. The idea is that the wavelets will resolve the wide range of space-time scales, and one of the techniques described herein will show how to model the interaction.

4 Competing small effects should be independent

It is characteristic of many interesting physical problems that there are several “small” parameters. In the Ginzburg-Landau equation (24) for the pattern evolution of convection near onset there is $R - R_c$, the amplitude A , the spatial derivative ∂_x , and potentially the along-roll spatial derivative ∂_y . In the flow of thin fluid films over a curved substrate, (17), there is the curvature of the substrate, the gradients of the free-surface, and potentially the Reynolds number, gravitational forcing, surface contamination, etc. The centre manifold approach enables a rational treatment of many and varied small effects, and then allows any consistent truncation when the model is applied.

4.1 Traditional scaling is restrictive

In the construction of dynamical models it is traditional to scale *a priori* all such small effects together in terms of a *single* small parameter, say ϵ . For example, to obtain the Ginzburg-Landau equation in convection: $R - R_c = \epsilon^2$, $A = \mathcal{O}(\epsilon)$, $\partial_X = \mathcal{O}(\epsilon)$, and $\partial_y = \mathcal{O}(\sqrt{\epsilon})$ [53, eqn.(4.6)]. For thin film flows, the substrate curvature $\kappa = \mathcal{O}(\epsilon^2)$, $\partial_x = \mathcal{O}(\epsilon)$ [155], and, if not neglected entirely, gravity $g = \mathcal{O}(\epsilon^2)$ unless the substrate is nearly all horizontal when $g = \mathcal{O}(\epsilon)$. This is done so that all the interesting dynamical effects occur at the one order in ϵ , namely the leading order. But, to give just one example, in the spatially extended system of thin film flow there may be regions where surface tension through substrate curvature is the dominant forcing, and other regions, where the substrate curvature is constant, in which gravity is the dominant forcing, and further the substrate could be nearly horizontal in some places and nearly vertical in others. Demanding that all interesting effects be scaled to appear at leading order is too restrictive. It is a “straight-jacket” that traditional techniques force upon us at the outset of the modelling.

The situation may be even worse. Sometimes the leading order model is structurally unstable, as in the Taylor model of dispersion (11) which to leading order in ∂_x is just the advection equation $\partial_t C = -U\partial_x C$. Higher order corrections, not appearing at leading order, are necessary to obtain a reasonable, structurally stable model. For simple problems, such as shear dispersion [168, Eqn.(17)] and propagating waves [129, p9] or convection rolls, the trick of transforming the TOE to a reference frame moving with an appropriate velocity, U in the case of shear dispersion and the group velocity in the case of waves, causes the low-order advection to disappear and the leading order in the model is then the higher-order terms needed for structural stability. However, in experiments one may have interacting waves or rolls travelling with different speeds or even in completely different directions. A moving reference frame cannot assist the analysis for these problems. *The traditional scaling paradigm, of using one small parameter to scale all others and then requiring all effects to appear at leading-order, is fundamentally flawed.*

4.2 A consistent flexibility

In contrast, centre manifold theory asserts, through Theorem 3, that one can approximate the shape of the centre manifold and the evolution thereon to any order in amplitude s . Competing small effects may appear at any order in the analysis, they need not just arise at leading order; after all the Taylor

series for $\exp(x)$ is $1 + x + x^2/2 + \dots$ to any convenient order. It is then consistent to include all terms up to a specific order in the model.

However, with parameters one may be considerably more flexible. For example, Theorem 4 shows that the model describing the pitchfork bifurcation in

$$\dot{x} = \epsilon x - xy, \quad \text{and} \quad \dot{y} = -y + x^2, \quad (26)$$

may be written variously as

$$\begin{aligned} \dot{x} &= \epsilon x - x^3 + \mathcal{O}(x^5 + \epsilon^{5/2}) \\ &= \epsilon x - (1 - 2\epsilon)x^3 + \mathcal{O}(x^5, \epsilon^2) \\ &= \epsilon x - (1 - 2\epsilon)x^3 - 2x^5 + \mathcal{O}(x^7 + \epsilon^{7/2}). \end{aligned}$$

These are all *consistent truncations* of the following multivariate asymptotic expansion

$$\begin{aligned} \dot{x} &= x\epsilon \\ &\quad -x^3(1 - 2\epsilon + 4\epsilon^2 - 8\epsilon^3 + 16\epsilon^4) \\ &\quad -x^5(2 - 16\epsilon + 88\epsilon^2 - 416\epsilon^3 + 1824\epsilon^4) \\ &\quad + \dots. \end{aligned}$$

As noticed by Coullet & Spiegel [44], this feature is very important. The centre manifold approach allows you to compute as many orders as you like, different orders in the amplitudes and the parameters (even different for the different parameters), and then *when you come to use the model, you may choose any consistent truncation that is appropriate for the particular realisation you wish to investigate*.

Such freedom is immensely valuable when there is any more than a couple of physical parameters in the problem.

5 The slow manifold is central

In a purely elastic body, elastic waves “ring” perpetually within the body. If these high frequency vibrations are ignored, then what is left is the relatively slow dynamics of rigid body motion. The flight of a ball is an example already mentioned in §1.

Muncaster and Cohen [118, 41] suggested the construction of the low-dimensional manifold of slow, rigid-body dynamics by neglecting the fast modes. The extremely simple example of the motion of a one-dimensional elastic body is discussed in [145, §2]. In contrast to the rapid collapse to

the centre manifold, the slow dynamics on the slow manifold form a low-dimensional model because they act as a “centre” for the fast oscillations of neighbouring trajectories, see Figure 8 for example. This principle of neglecting fast oscillations is completely equivalent to the *guiding centre* principle of Van Kampen [175], which is frequently invoked in plasma physics, see [21, 42, 29] for some recent work.

5.1 The linear basis—beam models

The construction of such a slow manifold uses exactly the same techniques as described earlier. It is based on the Approximation Theorem 3, or 4, with the distinction that the *slow manifold* (\mathcal{M}_0) has the *slow subspace* (\mathcal{E}_0) as its tangent space at the origin, instead of the centre subspace. The slow subspace being that space spanned by the eigenvectors and generalised eigenvectors of the *precisely zero* eigenvalues. For example, consider the dynamics of a long thin elastic beam. All the vibrations are fast except for the large-scale flexure, torsion and displacement of the beam. Applying the same ansatz of slowly-varying dependence along the beam [145], similar to that discussed in §§3.2 for dispersion, we may identify an 8-dimensional slow subspace in the cross-sectional elastic dynamics: displacements and velocities sideways (two), longitudinally and rotationally. Coupled with the assumed slow variations along the beam, the analysis holistically constructs dynamical models for the beam bending, torsion and stretching. In linear elasticity of a circular beam all these models decouple [145]. However, for a nonlinear or non-circular beam, the analysis naturally couples the dynamics. Computing the slow manifold to various asymptotic orders results in models equivalent to the range of models of classic beam theory. *The concept of a slow manifold is also useful in forming low-dimensional models of dynamics.*

You may have noted that all the specific examples of centre manifolds discussed in earlier sections have also been slow manifolds as they are based upon zero-eigenvalue modes. However, in this section I specifically focus on situations where the slow dynamics occurs among fast oscillations. That is, we take the spectrum of the linear dynamics to consist entirely of some zero eigenvalues and some purely imaginary eigenvalues.

5.2 Nonlinear problems—geostrophy

The above applications have so far been to notionally linear problems. The slow manifolds of nonlinear dynamics are also of interest. This is shown, for example, in the importance of the concept of geostrophy in atmospheric dynamics, see recent work in [23, 22]. Lorenz [102] introduced the five mode

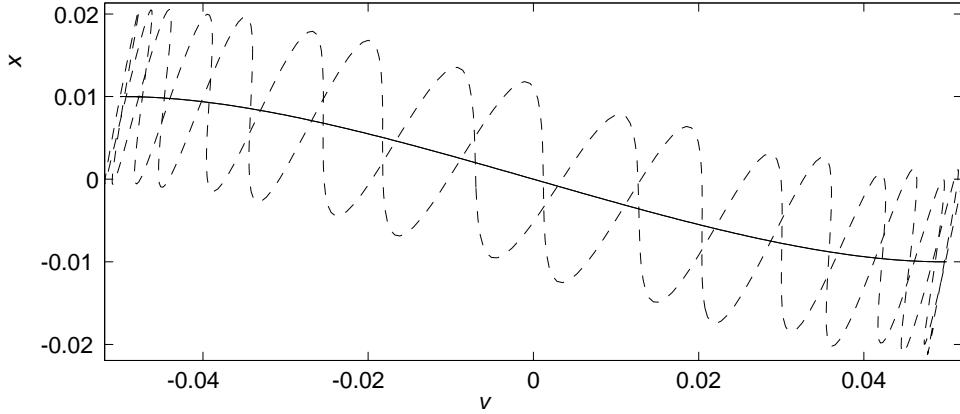


Figure 8: a comparison of the trajectories on (solid) and off (dashed) the slow manifold \mathcal{M}_0 for the Lorenz system (27). Observe that the fast oscillations off the slow manifold reasonably track the evolution on \mathcal{M}_0 .

dynamical system

$$\begin{aligned}
 \dot{u} &= -vw + bvz \\
 \dot{v} &= uw - buz \\
 \dot{w} &= -uv \\
 \dot{x} &= -z \\
 \dot{z} &= x + buv,
 \end{aligned} \tag{27}$$

to illuminate the nonlinear slow manifold of quasi-geostrophy. Observe that x and z oscillate quickly, while u , v and w evolve slowly, at least near the origin. A couple of trajectories of this system are plotted in Figure 8. The slow manifold of this 5-dimensional TOE is 3-dimensional based upon the three 0 eigenvalues associated with u , v and w in the linear dynamics. Theory by Sijbrand [160, §6–7] is relevant to the existence of a slow manifold for (27), namely

$$x = -buv + \mathcal{O}(s^4), \quad z = b(u^2 - v^2)w + \mathcal{O}(s^5),$$

where $s = |(u, v, w)|$, to some level of smoothness. Lorenz later argued [101] that truly slow dynamics do not exist in (27) because if it did then there must be an infinite number of singularities in the slow manifold. However, Cox & I [46] have shown that the singularities are exponentially weak and so are negligible for small enough amplitude flow. Recently, Bokhove & Shepherd [17], Boyd [20], Camassa [24], and Lorenz [103] have further elucidated the

structure and dynamics associated with the slow manifold of (27), and a connection with inertial manifolds has been made by Debussche & Temam [54, 55].

The concept of a slow manifold is generally applicable to Hamiltonian dynamical systems because of their typical spectrum. Examples are found in atmospheric dynamics [150], water waves [48, 127], and plasma physics [21, 128, 130, 179].

All the above issues are intriguing and deserve further study, but there is a further twist in such modelling: there is no relevance theorem for the low-dimensional dynamics on the slow manifold. There is no rigorous assurance that they do indeed model the TOE. Instead, Cox & I [46, 47] used normal forms, see §§6.2, to show that the dynamics on and off the slow manifold generally differ by an amount of $\mathcal{O}(r^2)$, where r measures the amplitude of the fast oscillations, it measures the distance off \mathcal{M}_0 . That is, there is some unavoidable slip between the model and the TOE. Thus, in general, *one can only expect a slow manifold model to be accurately predictive for a time $o(1/r^2)$* .

Lastly, I mention that Sijbrand [160], building upon work by Lyapunov, actually proves theorems about *sub-centre manifolds*. That is, manifolds based upon the eigenspace of a pair of pure imaginary modes; eigenvalues that are precisely 0 are a special case. His theorems are directly relevant to the existence and construction of nonlinear normal modes of oscillation as investigated by Shaw *et al* [158, 157, 159, 122]. It may be that such theorems also provide the basis for a justification of modulation equations describing the slow space-time evolution of the amplitude and phase of nonlinear dispersive waves. A simple example being the nonlinear Schrödinger equation usually derived using the method of multiple scales. However, a sub-centre manifold approach to the modulation and interaction of nonlinear waves has been elucidated in [144].

6 Initial conditions are long-lasting

Many low-dimensional models are used simply to explore the range of dynamical possibilities. For example, in control applications one wants to be assured that the specified control scheme will stabilise the system. However, many models are used to make definite predictions of later times given that the system starts from a given initial state—the forecast problem. So, suppose you know the initial state of the system for the TOE: what is the corresponding initial condition for the model?

For example, if a pollutant is released into a river from a given site, we

would wish to know what will be the dispersion of this specific cloud of pollutant. If the pollutant is released in the middle of a channel it will be initially carried downstream quicker than if it is released at the side; in the long-term the pollutant clouds will have different mean locations. The initial condition used for the model should be able to reflect such differences in the release. Appropriate initial conditions are also crucial in the derivation of correct boundary conditions, see §8.

Remarkably, this issue of providing the correct initial conditions for a low-dimensional dynamical model has received very little attention in the past. In many cases this is because attention has focussed on the typical dynamics inherent in a model. However, even when interested in making definitive forecasts, generally people have simply assumed that the provision either is according to the linear dynamics or is simply by the evaluation of the “amplitudes” in the model. That these assumptions are not always sound has been occasionally recognised in the phenomenon of “initial slip” [73, 80, 72, 174]. As developed in [139, 47] a useful procedure for determining correct initial conditions for a low-dimensional model is based on the geometric picture of a centre manifold in the state space.

6.1 Projecting initial conditions onto the model

The Relevance Theorem 2 assures us that there is indeed a particular solution of the low-dimensional model on \mathcal{M}_c which is approached exponentially by every trajectory of the TOE (provided the trajectory starts close enough to \mathcal{M}_c). This is illustrated in Figure 9 where trajectories of the dynamical system (1) from three different initial conditions all have the same long-term evolution *to a difference which decays exponentially quickly*. The modelling task is to find the projection P from any given initial state off \mathcal{M}_c , say \mathbf{u}_0 , onto a state on \mathcal{M}_c , say $\mathbf{v}(\mathbf{s}_0)$, so that the long-term evolution, from \mathbf{s}_0 in the model, will be the same to an exponentially small error. Some algebra finds this projection.

To be precise, the projection will be along the curved *isochronic manifolds*, or *isochrons* [181, 78], as shown in [139]. However, as in the example plotted in Figure 9, over a large part of the state space a linear approximation to the projection may be quite adequate; “linear” in distance away from \mathcal{M}_c , but varying with \mathbf{s} along \mathcal{M}_c . Such a linear projection is most easily defined by linearly independent normal vectors to the isochronic manifold at \mathcal{M}_c , such as $\mathbf{z} = (1, -x/(1 + 2x^2))$ shown in Figure 9. That is, we approximate by projecting along the tangent planes of the isochronic manifolds. Then the initial condition, \mathbf{s}_0 , for the model is found from the requirement that the

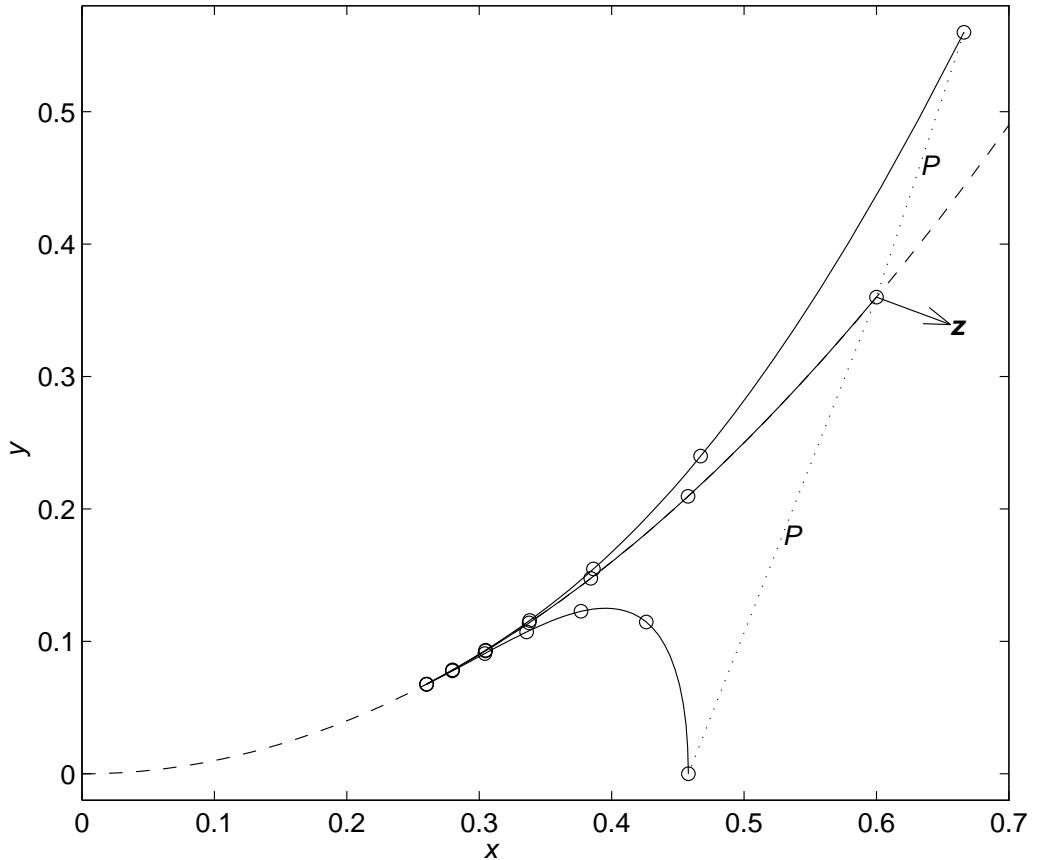


Figure 9: trajectories (solid) of (1) from three different initial conditions, all with the same long-term dynamics on the centre manifold \mathcal{M}_c (dashed) to an exponentially small error. The circles are plotted at $\Delta t = 1$ apart. The modelling issue is to find the projection P (dotted) from any given initial condition off \mathcal{M}_c onto one for the model on \mathcal{M}_c . P is described by its normal vector z at \mathcal{M}_c .

displacement is orthogonal to the normal vectors \mathbf{z}_j :

$$\langle \mathbf{z}_j, \mathbf{u}_0 - \mathbf{v}(\mathbf{s}_0) \rangle = 0, \quad (28)$$

where the angle brackets denote a suitable inner product. For an m -dimensional centre manifold within an n -dimensional dynamical system, the isochronic manifolds are of dimension $n - m$ (the state space “collapses” by this many dimensions), and so we need m linearly independent normal vectors \mathbf{z}_j to define this projection.

In the immediate vicinity of the fixed point at the origin of the general dynamical system (6), linear arguments suggest the vectors \mathbf{z}_j are eigenvectors, or generalised eigenvectors, of the adjoint eigen-problem

$$\mathcal{L}^\dagger \mathbf{z}_j = \mathbf{0}.$$

These give the correct projection onto \mathcal{E}_c under the linear dynamics. Under the nonlinear dynamics inherent upon the centre manifold \mathcal{M}_c , dynamical arguments given in [139] show that to find the projection vectors $\mathbf{z}_j(\mathbf{s})$ as a function of position on \mathcal{M}_c , we solve

$$\mathcal{D}\mathbf{z}_j - \sum_k \langle \mathcal{D}\mathbf{z}_j, \mathbf{e}_k \rangle \mathbf{z}_k = \mathbf{0} \quad \text{and} \quad \langle \mathbf{z}_j, \mathbf{e}_k \rangle = \delta_{jk}, \quad (29)$$

where \mathcal{D} encapsulates the dynamics of trajectories near \mathcal{M}_c as

$$\mathcal{D}\mathbf{z} = \frac{\partial \mathbf{z}}{\partial t} + \mathcal{J}^\dagger \mathbf{z},$$

in which the chain rule determines that

$$\frac{\partial \mathbf{z}}{\partial t} = \frac{\partial \mathbf{z}}{\partial \mathbf{s}} [\mathcal{G}\mathbf{s} + \mathbf{g}(\mathbf{s})],$$

and \mathcal{J}^\dagger is the adjoint of the Jacobian

$$\mathcal{J} = \mathcal{L} + \left. \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right|_{\mathcal{M}_c}.$$

As shown in [109, 110, 178], upon applying the formulae (28) and (29) to the dispersion of a contaminant in a channel or pipe, we accurately predict the different displacements of the mean concentration that occur from different release positions across the channel or pipe. The similar phenomena of a long-term variance deficit in the spread of the contaminant is also predicted. *With the geometric picture of centre manifolds we have created a mechanism to provide correct initial conditions for low-dimensional dynamical models.*

6.2 Normal forms also show the way

Commonly, as intimated at the start of §3, the normal form transformation is applied to the dynamics *after* reduction from the TOE to the centre manifold. The normal form then assisting in the classification of the dynamics. However, the normal form of the complete dynamics of the TOE clearly shows both the low-dimensional model (Elphick *et al* [62]) and the correct projection of initial conditions (Cox & I [47]).

For simplicity of discussion suppose that the dynamics of the TOE are given in the separated form (4–5). Then we may seek a nonlinear, but near identity, coordinate transformation

$$\mathbf{x} = \mathbf{X} + \mathbf{\Xi}(\mathbf{X}, \mathbf{Y}), \quad \text{and} \quad \mathbf{y} = \mathbf{Y} + \mathbf{\Psi}(\mathbf{X}, \mathbf{Y}),$$

such as that shown in Figure 10, so that the TOE is transformed to

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{y}) & \dot{\mathbf{X}} &= A\mathbf{X} + \mathbf{F}(\mathbf{X}) \\ \dot{\mathbf{y}} &= B\mathbf{y} + \mathbf{g}(\mathbf{x}, \mathbf{y}) & \dot{\mathbf{Y}} &= B\mathbf{Y} + G(\mathbf{X}, \mathbf{Y})\mathbf{Y} \end{aligned} \quad (30)$$

This is always possible given the pattern of eigenvalues for the existence of a centre manifold. For the example system (1), the nonlinear coordinate transformation

$$\begin{aligned} x &= X + XY + \frac{3}{2}XY^2 + \dots \\ y &= Y + X^2 + 2Y^2 + 4Y^3 + \dots, \end{aligned}$$

shown in Figure 10, separates the dynamics to

$$\dot{X} = -X^3, \quad \text{and} \quad \dot{Y} = -Y - 2X^2Y + \dots.$$

In the form (30), observe that $\mathbf{Y} = \mathbf{0}$ is clearly the invariant and exponentially attractive centre manifold. But also, both on and off \mathcal{M}_c , the \mathbf{X} evolution is completely independent of the decaying modes \mathbf{Y} . Hence all solutions of the TOE from initial conditions with the same \mathbf{X} have precisely the same \mathbf{X} evolution for all time, and thus they all tend to the same solution on the centre manifold. Thus the normal form shows that the projection of initial conditions onto the centre manifold is along constant \mathbf{X} . Constant \mathbf{X} are the isochronic manifolds.

The normal form transformation works its magic by decoupling the centre modes, \mathbf{X} , from the quickly decaying modes \mathbf{Y} . For slow manifold models which neglect fast oscillations, that is guiding centre models, a normal form coordinate transform will similarly exhibit the slow manifold as $\mathbf{Y} = \mathbf{0}$

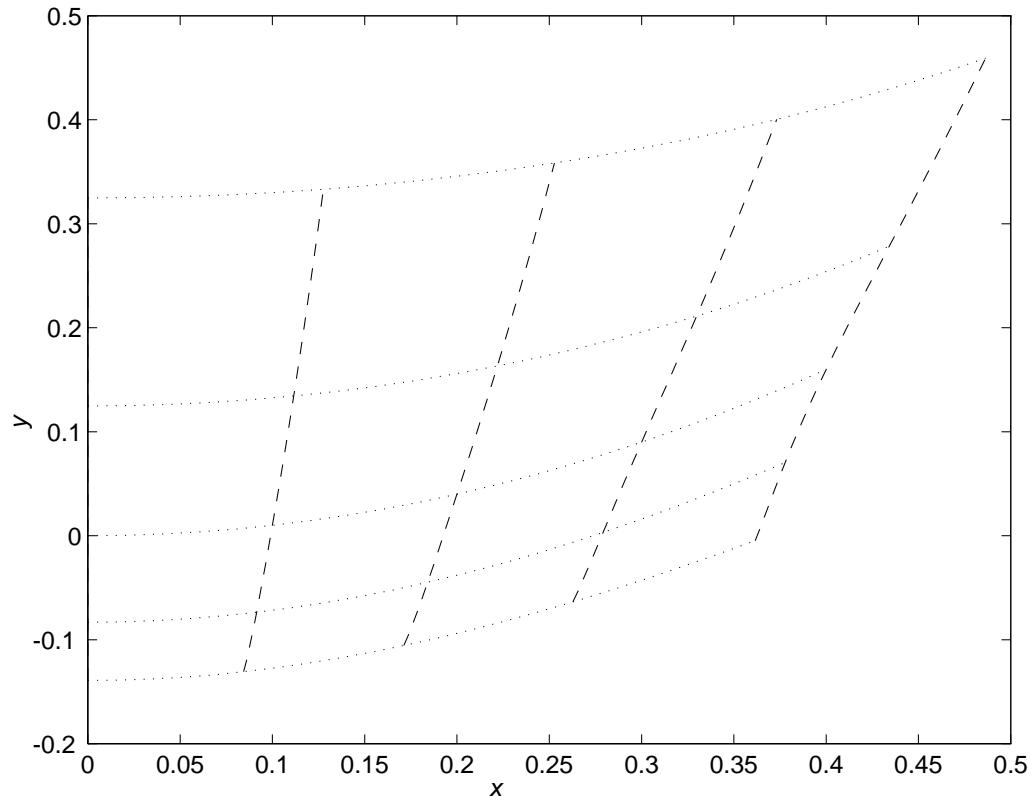


Figure 10: the normal form transformation of (1): the dashed lines are the constant X isochrons; the dotted lines are constant Y ; with $\Delta X = \Delta Y = 0.1$. The centre manifold is the dotted curve ($Y = 0$) emanating from the origin.

[46, 47]. The coordinate transform will also remove terms linear in \mathbf{Y} from $\dot{\mathbf{X}}$. However, in general, terms quadratic in \mathbf{Y} cannot be eliminated from $\dot{\mathbf{X}}$ —there is generally a resonant forcing of the slow modes, \mathbf{X} , by the fast oscillations or waves, \mathbf{Y} , which is quadratic in the oscillation amplitude. Thus, as mentioned in §5, the evolution of the TOE is generally slightly but unavoidably different to that of the slow model. An example would be the phenomenon of Stokes drift generated by any “fast” water waves superimposed upon the dynamics of large-scale currents. There are limitations to low-dimensional models based upon the concept of a slow manifold (or a sub-centre manifold).

Although the normal form of the TOE is a very useful conceptual tool, it does not provide a practical method for constructing a low-dimensional model. The reason is that it involves considerable wasted algebra in the transformation of the ultimately neglected stable and/or fast modes. If there are many stable mode, as usual in interesting physical systems, then the normal form transformation may be practically impossible, whereas methods based upon the Approximation Theorem 3 will be manageable.

7 Enforcing some surprises

So far we have implicitly restricted attention to unforced dynamical systems. The presence of small forcing in the TOE may be transformed into a forcing of the model. Discussed in the following subsections are the cases of deterministic forcing and of stochastic forcing or noise. The centre manifold formalism, coupled with the projection of initial conditions, permits an accurate modelling of the effects of the forcing. Remarkably, small forcing that otherwise would be neglected can have a large effect on the model’s dynamics.

7.1 Deterministic forcing

Consider the dynamical system (1) with the y mode forced by a steady effect ϵ :

$$\dot{x} = -xy \quad \text{and} \quad \dot{y} = -y + x^2 - 2y^2 - \epsilon. \quad (31)$$

As for the initial condition problem, a simple projection of the forcing onto \mathcal{E}_c would indicate that this particular forcing would have little influence on the low-dimensional model. However, this is not so. The correct model is that on the perturbed centre manifold $y \approx -\epsilon + (1 + 2\epsilon)x^2$ the evolution is

$$\dot{x} \approx \epsilon x - (1 + 2\epsilon)x^3, \quad (32)$$

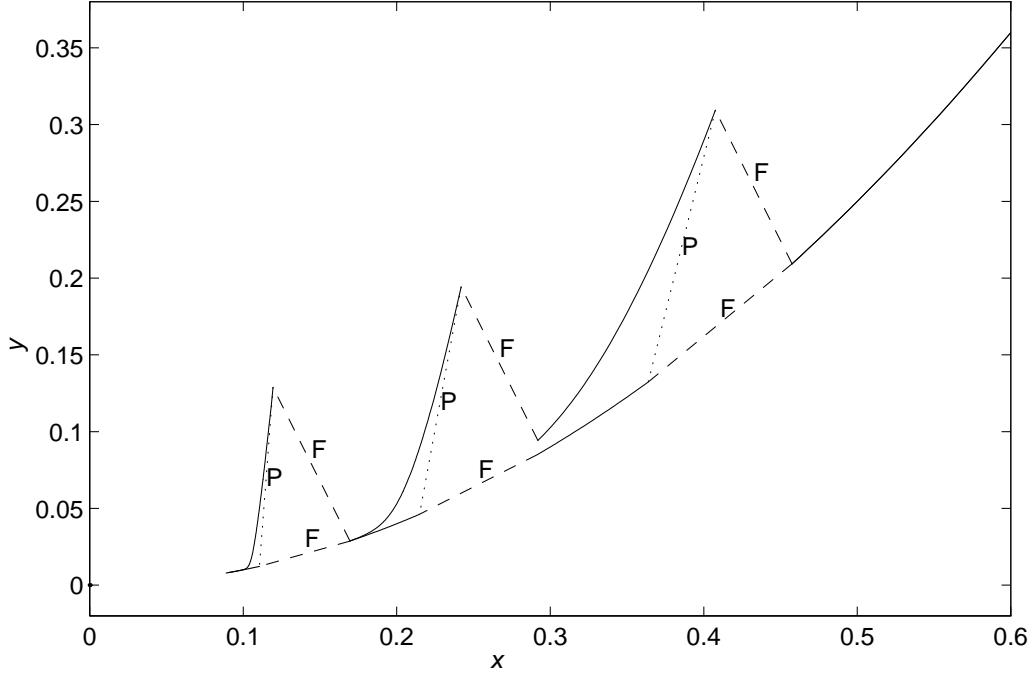


Figure 11: evolution of a dynamical system with discrete impulsive forcing and the corresponding evolution of its low-dimensional model on \mathcal{M}_c . The trajectories (solid) are “kicked” by each impulse (F , dashed). To maintain the correspondence between the exact solution and its model, the impulse has to be projected (P , dotted) along the isochronic manifolds onto \mathcal{M}_c to become an impulsive forcing for the model.

which exhibits a pitchfork bifurcation with two stable fixed points at $x \approx \sqrt{\epsilon}$. This is a large response in the model to a small forcing. Such a significant change in a model may be typical because the slow evolution on the centre manifold is sensitive to small influences.

Forcing may be incorporated into a model by two approaches. Firstly, if *the forcing is constant*, the TOE is autonomous, then a simple trick will suffice to put the dynamical system within the rigorous scope of centre manifold theory. For the above example, simply substitute $\epsilon = \delta^2$, say, and adjoin the trivial dynamical equation $\delta = 0$. Then the forcing becomes a nonlinear term in the dynamics, the spectrum shows a centre manifold parameterised by the modes x and δ , and the construction of the centre manifold and the evolution thereon leads to the forced model (32).

Secondly, *for time-dependent forcing* an argument developed in [139] will suffice. Suppose a general dynamical system (6) has a small applied forcing

$\mathbf{F}(t)$, namely

$$\dot{\mathbf{u}} = \mathcal{L}\mathbf{u} + \mathbf{f}(\mathbf{u}) + \mathbf{F}(t). \quad (33)$$

Imagine approximating the forcing by a sum of impulses, $\mathbf{F} = \sum_j \mathbf{F}_j \delta(t - t_j)$, at discrete times t_j . Then within each force free interval, the centre manifold is exponentially attractive, as shown in Figure 11, and the unforced model applies. Each impulse “kicks” the dynamical system off \mathcal{M}_c , see Figure 11. As in the initial condition problem, the evolution following such a “kick” exponentially quickly approaches the evolution of some particular solution on \mathcal{M}_c . This solution is also obtained by some particular impulse, found by projecting the impulse \mathbf{F}_j , applied to the low-dimensional model. The sequence of such projected impulses then approximates a continuous forcing of the model, say $\mathbf{G}(\mathbf{s}, t)$ in

$$\dot{\mathbf{s}} = \mathcal{G}\mathbf{s} + \mathbf{g}(\mathbf{s}) + \mathbf{G}(\mathbf{s}, t). \quad (34)$$

Letting

$$B_{ij} = \text{inverse of } \left\langle \mathbf{z}_i, \frac{\partial \mathbf{v}}{\partial s_j} \right\rangle,$$

the appropriate projection of the forcing of the TOE into a forcing of the model is

$$G_i = \sum_j B_{ij} \langle \mathbf{z}_j, \mathbf{F} \rangle. \quad (35)$$

This argument is accurate to errors $\mathcal{O}(F^2)$, it gives the linear projection of the forcing. Algebraic formula for higher order corrections, in F , have not yet been found.

For the example (31), $G = \epsilon x$ and hence the forced model is

$$\dot{x} \approx \epsilon x - x^3. \quad (36)$$

This model is apparently different to the earlier (32). The reason for the difference is that *implicitly* we have used a different parameterisation of the forced centre manifold. Cox & I [45] showed that there is extra freedom in parameterising a forcing because such forcing typically moves the evolution off the centre manifold into new regions of state space. However, we also showed that, in general, the forcing in the model then undesirably depends upon an integral over the previous history of the time-dependent forcing—a memory effect as also noted in §§3.2 for shear dispersion [162]. The *only* way to eliminate memory integrals in the dynamics of the model is to use the isochronic manifolds as a basis for the parameterisation off \mathcal{M}_c , as done implicitly in the argument above. *The geometric picture provided by centre manifold theory enables us to correctly incorporate forcing.*

In application to beam theory [145], for example, the correct treatment of forcing allows us to predict the “toothpaste effect”, that is, we predict the longitudinal extrusion of an elastic rod which results from a purely lateral compression!

7.2 Stochastic dynamical systems

In practice we may need to model dynamics in a noisy environment [172, e.g.]. Just one example of interest is the dispersion of a contaminant in a turbulent river—the turbulence can only reasonably be modelled by invoking stochastic factors. How do we construct low-dimensional models of stochastic differential equations (SDEs) using centre manifold ideas and techniques?

Currently there seem to be a number of largely disparate threads in the modelling of SDEs. Boxler, Arnold *et al* [19, 7, 8] prove theoretical results about low-dimensional centre manifolds of SDEs. However, the analysis is sophisticated and the results, as seen in examples, seem difficult to apply. Analysis of Fokker-Plank equations lead to “weak models” [94, 63, 61, 49], “weak” in the sense that detailed information about the noise is irretrievably lost to the model. Systematic adiabatic elimination [152, 153, 154] and normal form transformations [43, 120, 121] have also been suggested, but typically suffer from introducing many complicated noise processes into the dynamical model.

Inspired by the work on forcing described in the previous section, Chao & I [31, §3] realised that much of the complication may be removed from the noise in a low-dimensional model. On the long time-scale of the evolution on the centre manifold, the noise processes are essentially white and a little algebra, based upon the freedom to parameterise the forced centre manifold, explicitly shows this. However, such simplification only works for effects linear in the stochastic noise. Effects quadratic in the noise are more complicated. For example, stochastic resonance leads to *irreducible noise* processes appearing in the model, such as $w(t)$ from the (Stratonovich) SDE

$$dw = Z dW, \quad \text{and} \quad dZ = -\beta Z dt + dW, \quad (37)$$

for a given noise $W(t)$ in the TOE. Observe that Z is a coloured noise generated directly by W , then Z and W together induce the new noise w . Neither normal form nor centre manifold algebra can simplify the representation of the induced stochastic process $w(t)$. However, a centre manifold analysis of the corresponding Fokker-Plank equation [31, §4] suggests that we replace $w(t)$ by its long-term drift and fluctuation,

$$dw \approx \frac{1}{2} dt + \frac{1}{2\sqrt{\beta}} dW', \quad (38)$$

where $W'(t)$ denotes a *new* and *independent* noise. In essence, the nonlinear machinations in (37) extracts information about W that cannot be obtained by sampling W over large times, information which is thus independent and can only be represented on the long time-scales of the model by a new noise process.

Much further research is needed to apply these ideas and provide theoretical support.

8 At boundaries

Models expressed as partial differential equations, such as those for dispersion in a river and for beam theory, require boundary conditions at the limits of the domain, the inlet and outlet [163] or the ends of the beam respectively. Such models as these are typically derived through a slowly-varying approximation under the assumption that the domain of interest is arbitrarily large as discussed in §3. However, typical physical situations of interest possess finite domains. The issue is: what are the correct boundary conditions to be used at the edge of the domain for such model equations? The provision of correct boundary conditions are sometimes crucial—in large aspect-ratio convection the boundary drives the entire nature of the long-term dynamics, see [126, p444] or [75, §3].

Relatively little work has been done on determining correct boundary conditions. When interested in generic dynamical behaviour one typically just uses periodic boundary conditions, [95, 87, 3] for example. Alternatively, one may only allow boundaries in the original problem which will fit neatly into the scheme of the asymptotic approximation, [33, 156, 145] for example. These two choices are forced by the inability of primitive asymptotic schemes, such as the method of multiple scales, to embrace the presence of typical physical boundaries. Alternatively, in realistic physical problems heuristic arguments may supply approximate boundary conditions for the model, but are they mathematically justifiable? Examples are the boundary conditions used with beam theory for the idealisations of free, fixed or pinned ends. Cross and others [51, 52, 106] have used matched asymptotics to derive boundary conditions for two-dimensional convective rolls; however, their analysis is linear near the boundary, and yet needs to be nonlinear in the interior. More attention needs to be given to this important aspect of modelling.

Some special invariant manifolds (§§2.1) in conjunction with the correct projection of initial conditions (§6) provide a route to determine correct boundary conditions [142]. Boundary conditions are provided by two

separate and complementary arguments; both relying on the same trick of investigating the *spatial evolution* away from the boundary into the interior (also mentioned in §3), given also a slow time evolution. In essence, we probe the dynamics of the boundary layers at either end of the domain and how they merge into the slowly-varying interior solution. The dominant terms in the boundary conditions typically agree with those obtained through simple physical arguments. However, refined models of higher order require subtle corrections to the previously-deduced boundary conditions, and also require the provision of additional boundary conditions to eliminate unphysical predictions and to form a complete model.

8.1 The physical boundary layer

In the TOE with slow time evolution, the spatial evolution away from a boundary consists of: exponentially decaying “stable” modes; exponentially growing modes which are not seen as the far distant boundary removes them; and nearly neutral modes which correspond to those of the interior model. Thus near any boundary, because of the dynamics inherent in the differential equations of the TOE, the system must lie in the centre-stable manifold (to an exponentially small error in the length L of the domain). This centre-stable manifold may be constructed.

As an illustration of the ideas, consider the shear dispersion problem shown in Figure 4. If the time dependence is negligible, solutions $c'(y)e^{\lambda x}$ exist where

$$\frac{d^2c'}{dy^2} + (\delta\lambda^2 - u(y)\lambda)c' \approx 0. \quad (39)$$

There exists solutions:

- $\lambda = 0$, c' constant, corresponding to the critical mode of the slowly-varying centre manifold model;
- modes with negative eigenvalues, approximately $\lambda = -3.414$, $\lambda = -12.25$, etc, describing how the cross-stream diffusion, d^2c'/dy^2 , exponentially smooths out details of the inlet concentration as it is advected downstream by the profile $u(y)$;
- and modes with large positive eigenvalues, $\lambda \propto 1/\delta$, allowing upstream diffusion to accommodate the interior concentration to any particular outlet condition.

The centre-stable manifold near the inlet is approximately the space spanned by the critical mode with $\lambda = 0$ together with the modes of negative λ .

Conversely, at the outlet and viewing the dynamics towards negative x , the centre-stable manifold is approximately the span of the critical mode together with the modes of positive λ .

A boundary condition must be given as part of the TOE. Such a boundary condition intersects the constructed centre-stable manifold to form the set of possible states that may hold at the boundary. These states form a set of possible “initial conditions” for the spatial evolution away from the boundary. The states in the intersection are then projected onto the centre manifold (of the spatial evolution) using the initial condition arguments described in §6 to give a set of allowable states for the centre manifold model at the boundary. These determine some boundary conditions for the low-dimensional model.

For the example of shear dispersion in a pipe or channel, these arguments generally provide one inlet condition for the Taylor model (11), but provide no constraint at the outlet [142, §2, §3.1] because the centre-stable manifold at the outlet is of one higher dimension, thus has a larger intersection with the exit boundary conditions, and hence does not restrict the states of the model at the outlet.

8.2 The boundary layer of the model

As noted for Taylor’s model of dispersion, some low-dimensional models need more boundary conditions than the previous argument supplies. Such boundary conditions come from considering the dynamics of the model in the boundary layers at either end of the domain.

Given slow variations in time, the spatial evolution of the *model* into the interior consists of: critical modes which correspond to those of the low-dimensional model seen in the interior; unphysical exponentially decaying modes which do not penetrate far into the interior; and unphysical exponentially growing modes which the far boundary must remove. We require that there be no unphysical stable modes near a boundary [142, §3.2] as otherwise there will be unsightly and unphysical transients in the model’s solutions. Thus the system must lie in the centre-unstable manifold of the model, which we may construct. The fact that any solution actually lies in the centre manifold is assured by the far boundary because a boundary’s unstable modes correspond to the far boundary’s removed stable modes. This requirement provides sufficient additional boundary conditions [142, §3.3].

For example, in the Taylor model (11) and for negligible time variations, solutions $e^{\lambda x}$ exist with $\lambda \approx 0$ and $\lambda \approx U/D$. At the outlet, the stable mode, approximately $e^{Ux/D}$, is removed by enforcing a condition that the system is on the centre-unstable manifold (here just the centre manifold as there is no unstable mode). This provides a boundary condition at the exit to

complement the entry boundary condition from the previous argument. The argument here does not provide an entry boundary condition as there is no stable mode in the Taylor model at the inlet.

These two arguments together very neatly provide boundary conditions for models with one slowly-varying spatial dimension and slow variations in time. Outstanding is a resolution of the apparent differences between this approach and the more specialised approach employed for beams in [145], and the provision of boundary conditions for oscillatory critical modes. Also outstanding, but of considerable interest, is the issue of what to do for models involving *two* spatial dimensions, such as models of thin, elastic plates and of planform evolution in convection.

9 Computer algebra handles the details

Computing details of the shape of a centre manifold and the evolution thereon may involve copious algebra. A leading order approximation is often tractable by hand, but when this approximation is not structurally stable then the requisite higher-order corrections are typically tedious to determine. Extremely high-order calculations, for example to determine the spatial resolution of dispersion §§3.2 or beam theory [145], are grossly tedious. Computer algebra may reduce the human labour involved. The challenge is to develop algorithms which are simple to reliably implement.

9.1 Explicit series expansions

Various computer algebra packages have been used to compute centre manifolds, such as MAPLE [14], REDUCE [69] and MACSYMA [135], and also normal forms [135, 68].

In problems with centre manifolds of small dimension, 1, 2 or 3 say, the usual approach is simply to seek an explicit multinomial form. One then substitutes into the governing equations, gathers like terms, and solves for the unknown coefficients. See for example Rand & Armbruster's [134, pp27–34] MACSYMA code or Freire *et al*'s [69] REDUCE code for constructing the centre manifold of a finite-dimensional dynamical system. But this approach fails for centre manifolds of higher dimension, especially for the “infinite” dimensional centre manifolds associated with spatio-temporal models. For example, in lubrication models of thin film flow (§3.3) there are 5 basic terms of the leading, fourth-order, namely η_x^4 , $\eta_x^2\eta_{xx}$, η_{xx}^2 , $\eta_x\eta_{xxx}$ and η_{xxxx} , each of these possibly combined with the multiplication by an arbitrary power of η . *It very soon becomes impractical to code a complete sum of general terms.*

Another approach is to select one or two “ordering parameters” that measure the size of all nonlinearities and all other small effects. Then express the model explicitly in terms of asymptotic sums in these ordering parameters with functional coefficients which are to be determined [147, 108, e.g.]; the form of the unknown coefficients are found in the working, a general form need not be explicitly written down beforehand. To reduce the dynamics onto the centre manifold, one then has to substitute the asymptotic sums into the governing equations, reorder the summations, rearrange to extract dominant terms, and evaluate the expressions. While perfectly acceptable when done correctly, it leads to formidable working which obscures the construction of a model. Further, such asymptotic expansions, in common with the method of multiple scales, reinforces the notion that careful balancing of the “order” of small effects are necessary in the *construction* of a model rather than in its *use* in some situation (§4).

9.2 Implicit approximation

Recently I proposed [148] an iterative method, based upon the residuals of the governing differential equations of the TOE, as shown schematically in Figure 12. The evaluation of the residuals is a routine algebraic task which may be easily done using computer algebra by simply coding the governing differential equations; it replaces the whole messy detail of the manipulation of asymptotic expansions (e.g. [44, §5.4]). In the same spirit as comments made by Barton & Fitch [11, §2.3] in 1972, the aim of this proposed approach is to minimise human time by using a novel algorithm which is simply and reliably implemented in computer algebra, albeit with inefficiencies in the use of computer resources.

Given a generic dynamical system (6) we seek a centre manifold $\mathbf{u} = \mathbf{v}(\mathbf{s})$ on which $\dot{\mathbf{s}} = \mathcal{G}\mathbf{s} + \mathbf{g}(\mathbf{s})$. As in §§2.2.3, if ϕ and ψ approximate \mathbf{v} and $\dot{\mathbf{s}}$ respectively, then I argue [148] that corrections, ϕ' and ψ' , may be found by solving the *homological equation*

$$\mathcal{L}\phi' - \frac{\partial\phi'}{\partial\mathbf{s}}\mathcal{G}\mathbf{s} - \mathcal{V}\psi' = \mathbf{R}(\phi, \psi), \quad (40)$$

where $\mathcal{V} = \partial\phi/\partial\mathbf{s}|_{\mathbf{0}}$, and recall that \mathbf{R} is the residual of the TOE. Generally this produces linear convergence to \mathcal{M}_c and the low-dimensional evolution, linear in the sense that each iteration corrects the next order or two of the expressions. The algorithm for the computer algebra derivation of low-dimensional models is relatively: simple to implement, because the computation of the residual is via a *direct* coding of the governing differential

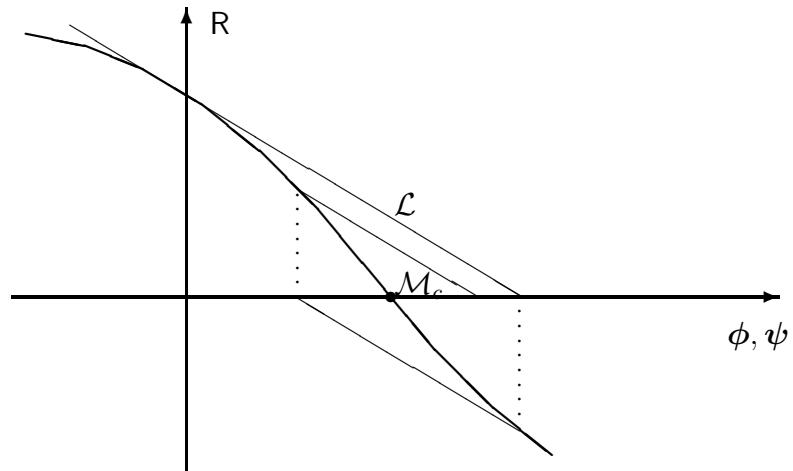


Figure 12: schematic diagram of an iteration to determine the centre manifold. It depicts the linear operator, \mathcal{L} , at the fixed point at the origin guiding corrections to the description, \mathbf{v} & \mathbf{G} , based upon the residuals of the TOE.

equations; flexible because many changes to the TOE can be simply accommodated by updating the residual computation; and reliable, because it relies upon the actual residual going to zero—any error is picked up by a lack of convergence.

This algorithm has already been used to develop models of thin film fluid dynamics upon curved substrates with gravitational and inertial effects [146, 149]. A currently outstanding issue is developing a companion iteration scheme to determine the correct projection of initial conditions (§6) and forcing (§7).

10 Conclusion

We have looked at geometric based arguments about how to form accurate and reliable low-dimensional models of dynamical systems. The aim is to take a so-called “Theory Of Everything” (TOE), a complete but far too detailed description of the system, and produce a “coarse grained” model that captures all the dynamics, and little else, that are of interest in a particular application. Although the geometric arguments are supported centre manifold theory, we have seen that many applications go beyond the current rigorous support.

This approach to modelling has many advantages:

- the signature of the model is straightforwardly derived from linearisation (§§2.2.1, §5), but one can be inventive (§§2.3, §§3.3, §§3.4);
- the model may be systematically refined (§§§2.2.3, §§2.3, §§3.2), perhaps using computer algebra to handle most of the details (§9);
- one may be very flexible about the relative magnitude of “small” parameters, resulting in a model which is later justifiably tuned to specific applications (§4);
- it provides correct initial conditions for forecasts (§6), initial conditions that guarantee fidelity between the predictions of the model and the long-term evolution of the full dynamics;
- may account for smooth or stochastic forcing (§7);
- establishes an approach to determine correct boundary conditions for the models (§8).

How to decide upon an accurate TOE on which to base the analysis is another problem—a model is only guaranteed if it is based upon an accurate description of the dynamics.

References

- [1] A. Afendikov and A. Mielke. A spatial center manifold approach to a hydrodynamical problem with $o(2)$ symmetry. In *Dynamics, bifurcation and symmetry (Cargese, 1993)*, volume 437 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 1–10. Kluwer Acad. Publ., Dordrecht, 1994.
- [2] M. Aouadi. Convection in double-diffusion with rigid boundaries. *C. R. Acad. Sci. Paris Ser. I Math.*, 316(4):393–397, 1993.
- [3] D. Armbruster, J. Guckenheimer, and P. Holmes. Kuramoto sivashinsky dynamics on the centre-unstable manifold. *Siam J Appl Math*, 49:676–691, 1989.
- [4] A. Arneodo, P.H. Coullet, and E.A. Spiegel. The dynamics of triple convection. *Geophys. Astro. Fluid Dyn.*, 31:1–48, 1985.
- [5] A. Arneodo, P.H. Coullet, E.A. Spiegel, and C. Tresser. Asymptotic chaos. *Physica D*, 14:327–347, 1985.
- [6] A. Arneodo and O. Thual. Direct numerical simulations of a triple convection problem versus normal form predictions. *Physic Letters A*, 109:367–372, 1985.
- [7] L. Arnold and P. Boxler. Elimination of fast variables in the presence of noise: stochastic center manifold theory. In *Synergetics, order and chaos (Madrid, 1987)*, pages 671–684. World Sci. Publishing, Teaneck, NJ, 1988.

- [8] L. Arnold and K. Xu. Simultaneous normal form and center manifold reduction for random differential equations. In *International Conference on Differential Equations, Vol. 1, 2 (Barcelona, 1991)*, pages 68–80. World Sci. Publishing, River Edge, NJ, 1993.
- [9] P. Ashwin, I. Moroz, and M. Roberts. Bifurcations of stationary, standing and travelling waves in triply diffusive convection. *Phys. D*, 81(4):374–397, 1995.
- [10] V. Balakotaiah and H.C. Chang. Dispersion of chemical solutes in chromatographs and reactors. *Phil Trans R Soc Lond A*, 351:39–75, 1995.
- [11] D. Barton and J.P. Fitch. Applications of algebraic manipulative programs in physics. *Rep. on Prog. Phys.*, 35:235–314, 1972.
- [12] R.A. Becker and C. Foias. The local bifurcation of ramsey equilibrium. *Econom. Theory*, 4(5):719–744, 1994.
- [13] S. Behtash and S. Sastry. Stabilization of nonlinear systems with uncontrollable linearization. In *Dynamical systems approaches to nonlinear problems in systems and circuits (Henniker, NH, 1986)*, pages 242–255. SIAM, Philadelphia, PA, 1988.
- [14] J. Belair and S.A. Campbell. Stability and bifurcations of equilibria in a multiple-delayed differential equation. *SIAM J. Appl. Math.*, 54(5):1402–1424, 1994.
- [15] E. Boe and H.C. Chang. Dynamics of delayed systems under feedback control. *Chem. Eng. Sci.*, 44:1281–1294, 1989.
- [16] E. Boe and H.C. Chang. Transition to chaos from a two-torus in a delayed feedback system. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 1(1):67–81, 1991.
- [17] O. Bokhove and T.G. Shepherd. On the hamiltonian balanced dynamics and the slowest invariant manifold. *J Atmos Sci*, 53(2):276–297, 1996.
- [18] W.M. Boothby and R. Marino. The center manifold theorem in feedback stabilization of planar, single input systems. *Control Theory Adv. Tech.*, 6(4):517–532, 1990.
- [19] P. Boxler. A stochastic version of the centre manifold theorem. *Probab. Th. Rel. Fields*, 83:509–545, 1989.
- [20] J.P. Boyd. The slow manifold of a five-mode model. *J. Atmospheric Sci.*, 51(8):1057–1064, 1994.
- [21] A.J. Brizard. Nonlinear gyrokinetic vlasov equation for toroidally rotating axisymmetric tokamaks. *Phys. Plasmas*, 2(2):459–471, 1995.
- [22] G. Brunet. On global nonlinear geostrophic balance of an isolated distribution of vorticity using a new time invariant. *J. Atmospheric Sci.*, 50(14):2304–2309, 1993.
- [23] A.B.G. Bush, J.C. McWilliams, and W.R. Peltier. Origins and evolution of imbalance in synoptic-scale baroclinic wave life cycles. *J. Atmospheric Sci.*, 52(8):1051–1069, 1995.
- [24] R. Camassa. On the geometry of an atmospheric slow manifold. *Physica D*, 84:357–397, 1995.

- [25] S.A. Campbell, J. Belair, T. Ohira, and J. Milton. Limit cycles, tori, and complex dynamics in a second-order differential equation with delayed negative feedback. *J. Dynam. Differential Equations*, 7(1):213–236, 1995.
- [26] J. Carr. *Applications Of Centre Manifold Theory*, volume 35 of *Applied Math Sci*. Springer-Verlag, 1981.
- [27] J. Carr and R.G. Muncaster. The application of centre manifold theory to amplitude expansions. ii. infinite dimensional problems. *J. Diff. Eqns*, 50:280–288, 1983.
- [28] J. Carr and R.G. Muncaster. The application of centre manifold theory to amplitude expansions.i. ordinary differential equations. *J. Diff. Eqns.*, 50:260–279, 1983.
- [29] J.R. Cary and S.G. Shasharina. Probability of orbit transition in asymmetric toroidal plasma. *Phys. Fluids B*, 5(7):2098–2121, 1993.
- [30] H.C. Chang. Wave evolution on a falling film. *Annu. Rev. Fluid Mech.*, 26:103–136, 1994.
- [31] Xu Chao and A.J. Roberts. On the low-dimensional modelling of Stratonovich stochastic differential equations. *Physica A*, 225:62–80, 1996.
- [32] C.J. Chapman, S. Childress, and M.R.E. Proctor. Long wavelength thermal convection between non-conducting boundaries. *Earth & Planet. Sci. Lett.*, 51:342–369, 1980.
- [33] C.J. Chapman and M.R.E. Proctor. Nonlinear rayleigh-benard convection between poorly conducting boundaries. *J. Fluid Mech.*, 101:759–782, 1980.
- [34] C.C. Chen and H.C. Chang. Accelerated disturbance damping of an unknown disturbed system by nonlinear feedback. *Aiche J*, 38:1461–1476, 1992.
- [35] A.H.D. Cheng, C.Y. Yang, K. Hackl, and M.J. Chajes. Stability, bifurcation and chaos of nonlinear structures with control. ii. nonautonomous case. *Internat. J. Non Linear Mech.*, 28(5):549–565, 1993.
- [36] M. Cheng and H.C. Chang. Subharmonic instabilities of finite amplitude monochromatic waves. *Phys Fluids A*, 4:505–523, 1992.
- [37] C. Chiarella. *The elements of a nonlinear theory of economic dynamics*, volume 343 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 1990.
- [38] P. Chossat, Y. Demay, and G. Iooss. Competition of azimuthal modes and quasiperiodic flows in the couette-taylor problem. In *Dynamics of infinite-dimensional systems (Lisbon, 1986)*, volume 37 of *NATO Adv. Sci. Inst. Ser. F: Comput. Systems Sci.*, pages 45–56. Springer, Berlin-New York, 1987.
- [39] P. Chossat and M. Golubitsky. Hopf bifurcation in the presence of symmetry, center manifold and liapunov-schmidt reduction. In *Oscillations, bifurcation and chaos (Toronto, Ont., 1986)*, volume 8 of *CMS Conf. Proc.*, pages 343–352. Amer. Math. Soc., Providence, RI, 1987.
- [40] S.N. Chow and K.N. Lu. C^k centre unstable manifolds. *Proc Roy Soc Edin A*, 108:303–320, 1988.

- [41] H. Cohen and R.G. Muncaster. The theory of psuedo-rigid bodies. *Springer Tracts In Natural Philosophy*, 33, 1988.
- [42] D.R. Cook, A.N. Kaufman, A.J. Brizard, H. Ye, and E.R. Tracy. Two-dimensional reflection of magnetosonic radiation by gyroballistic waves: an analytic theory. *Phys. Lett. A*, 178(5-6):413–418, 1993.
- [43] P.H. Coullet, C. Elphick, and E. Tirapegui. Normal form of a hopf bifurcation with noise. *Physics Letts*, 111A(6):277–282, 1985.
- [44] P.H. Coullet and E.A. Spiegel. Amplitude equations for systems with competing instabilities. *SIAM J. Appl. Math.*, 43:776–821, 1983.
- [45] S.M. Cox and A.J. Roberts. Centre manifolds of forced dynamical systems. *J. Austral. Math. Soc. B*, 32:401–436, 1991.
- [46] S.M. Cox and A.J. Roberts. Initialisation and the quasi-geostrophic slow manifold. Submitted, 1994.
- [47] S.M. Cox and A.J. Roberts. Initial conditions for models of dynamical systems. *Physica D*, 85:126–141, 1995.
- [48] W. Craig and M.D. Groves. Hamiltonian long-wave approximations to the water-wave problem. *Wave motion*, 19(4):367, 1994.
- [49] J.D. Crawford. Amplitude expansions for instabilities in populations of globally-coupled oscillators. *J. Statist. Phys.*, 74(5-6):1047–1084, 1994.
- [50] J.D. Crawford and E. Knobloch. Symmetry and symmetry-breaking bifurcations in fluid dynamics. *Annu Rev Fluid Mech*, 23:341–387, 1991.
- [51] M.C. Cross. Boundary conditions on the envelope function of convective rolls close to onset. *Phys Fluids*, 25:936–941, 1982.
- [52] M.C. Cross, P.G. Daniels, P.C. Hohenberg, and E.D. Siggia. Phase winding solutions in a finite container above the convective threshold. *J Fluid Mech*, 127:155–183, 1983.
- [53] M.C. Cross and P.C. Hohenberg. Pattern formation outside of equilibrium. *Rev Mod Phys*, 65(3):851–1112, 1993.
- [54] A. Debussche and R. Temam. Inertial manifolds and slow manifolds. *Appl. Math. Lett.*, 4(4):73–76, 1991.
- [55] A. Debussche and R. Temam. Inertial manifolds and the slow manifolds in meteorology. *Differential Integral Equations*, 4(5):897–931, 1991.
- [56] M.C. Depassier and E.A. Spiegel. Convection with heat flux prescribed on the boundaries. i the effect of temperature dependence of material properties. *Geophys Astrophys Fluid Dyn*, 21:167–188, 1982.
- [57] D.R. Durran. Improving the anelastic approximation. *J. Atmos. Sci.*, 46:1453–1461, 1989.
- [58] T.A.W. Dwyer. Slew-induced deformation shaping on slow integral manifolds. In *Dynamics and control of multibody systems (Brunswick, ME, 1988)*, volume 97 of *Contemp. Math.*, pages 97–113. Amer. Math. Soc., Providence, RI, 1989.

- [59] W. Eckhaus. The ginzburg-landau manifold is an attractor. *J. Nonlinear Sci.*, 3:329–348, 1993.
- [60] M.P. Edwards. Classical symmetry reductions of nonlinear diffusion–convection equations. *Phys Letts A*, 190:149, 1994.
- [61] C. Elphick, M. Jeanneret, and E. Tirapegui. Comment on bifurcations in fluctuating systems. *J. Statist. Phys.*, 48(3-4):925–933, 1987.
- [62] C. Elphick, E. Tirapegui, M.E. Brachet, P. Coullet, and G. Iooss. A simple global characterisation for mormal forms of singular vector fields. *Physica D*, 29:95–127, 1987.
- [63] A. Fernandez. Constraints on the time-reversible liouville equation in order to derive a stochastic order-parameter equation at the onset of a convective roll pattern. *J. Phys. A*, 20(12):L763–L767, 1987.
- [64] C. Foias, M.S. Jolly, I.G. Kevrekidis, G.R. Sell, and E.S. Titi. On the computation of inertial manifolds. *Phys. Lett. A*, 131:433–436, 1988.
- [65] C. Foias, M.S. Jolly, I.G. Kevrekidis, and E.S. Titi. On some dissipative fully discrete nonlinear galerkin schemes for the kuramoto–sivashinsky equation. *Phys Lett A*, 186(1):87, 1994.
- [66] C. Foias, G.R. Sell, and R. Temam. Inertial manifolds for nonlinear evolution equations. *J. Diff. Eqn.*, 73:309–353, 1988.
- [67] C. Foias, G.R. Sell, and R. Teman. Inertial manifolds for dissipative differential equations. *Comptes Rendus Serie I*, 301:139–141, 1985.
- [68] E. Freire, E. Gamero, and E. Ponce. Symbolic computation and bifurcation methods. In *Continuation and bifurcations: numerical techniques and applications (Leuven, 1989)*, volume 313 of *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, pages 105–122. Kluwer Acad. Publ., Dordrecht, 1990.
- [69] E. Freire, E. Gamero, E. Ponce, and L.G. Franquelo. An algorithm for symbolic computation of center manifolds. In *Symbolic and algebraic computation (Rome, 1988)*, volume 358 of *Lecture Notes in Comput. Sci.*, pages 218–230. Springer, Berlin-New York, 1989.
- [70] J. Fujimura. Methods of centre manifold multiple scales in the theory of nonlinear stability for fluid motions. *Proc Roy Soc Lond A*, 434:719–733, 1991.
- [71] Th. Gallay. A center-stable manifold theorem for differential equations in banach spaces. *Commun. Math. Phys.*, 152:249–268, 1993.
- [72] U. Geigenmüller, U.M. Titulaer, and B.U. Felderhof. Systematic elimination of fast variables in linear systems. *Physica A*, 119:41–52, 1983.
- [73] H. Grad. Asymptotic theory of the boltzmann equation. *Phys. Fluids*, 6:147–181, 1963.
- [74] M.D. Graham and P.H. Steen. Strongly interacting traveling waves and quasiperiodic dynamics in porous medium convection. *Phys. D*, 54(4):331–350, 1992.
- [75] H.S. Greenside. Spatiotemporal chaos in large systems: the scaling of complexity with size. <http://xxx.lanl.gov/chao-dyn/9612004>, December 1996.

- [76] H.S. Greenside and W.M. Coughran. Nonlinear pattern-formation near the onset of rayleigh-benard convection. *Phys Rev A*, 30:398–428, 1984.
- [77] J. Grzedzinski. Flutter analysis of a two-dimensional airfoil with nonlinear springs based on center-manifold reduction. *Arch. Mech. (Arch. Mech. Stos.)*, 46(5):735–755, 1994.
- [78] J. Guckenheimer. Isochrons and phaseless sets. *J. Math. Biol.*, 1:259–273, 1975.
- [79] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 1983.
- [80] F. Haake and M. Lewenstein. Adiabatic drag and intial slip in random processes. *Phys. Rev. A*, 28:3060–3612, 1983.
- [81] H. Haken. *Synergetics, An introduction*. Springer, Berlin, 1983.
- [82] J.K. Hale. *Dynamics and delays*, volume 1475 of *Lecture Notes in Math*. Springer, Berlin, 1991.
- [83] J.K. Hale and H. Kocak. *Dynamics and bifurcations*, volume 3 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1991.
- [84] A. Hill and I. Stewart. 3-mode interactions with $o(2)$ symmetry and a model for taylor-couette flow. *Dynamics Stability Systems*, 6(4):267–339, 1991.
- [85] M. Hărăguș. Model equations for water waves in the presence of surface tension. *preprint*, 1995.
- [86] M. Hărăguș. Reduction of pdes on unbounded domains application: unsteady water waves problem. *preprint*, 1995.
- [87] J.M. Hyman, B. Nicolaenko, and S. Zaleski. Order and complexity in the kuramoto-sivashinsky model of weakly turbulent interfaces. *Physica D*, 23:265–292, 1986.
- [88] G. Iooss and M. Adelmeyer. *Topics in Bifurcation Theory*. World Sci, 1992.
- [89] G. Iooss and J. Los. Bifurcation of spatially quasi-periodic solutions in hydrodynamic stability problems. *Nonlinearity*, 3(3):851–871, 1990.
- [90] G. Iooss and A. Mielke. Bifurcating time-periodic solutions of navier-stokes equations in infinite cylinders. *J. Nonlinear Sci.*, 1(1):107–146, 1991.
- [91] G. Iooss, A. Mielke, and Y. Demay. Mathematical justification of steady ginzburg-landau equation starting from navier-stokes. In *New trends in nonlinear dynamics and pattern-forming phenomena (Cargese, 1988)*, volume 237 of *NATO Adv. Sci. Inst. Ser. B Phys.*, pages 275–286. Plenum, New York, 1990.
- [92] A. Isidori. *Nonlinear control systems*. Communication and Control Engineering Series. Springer-Verlag, Berlin-New York, 1989.
- [93] K. Khorasani. Feedback equivalence for a class of nonlinear singularly perturbed systems. *IEEE Trans. Automat. Control*, 35(12):1359–1363, 1990.
- [94] E. Knobloch and K.A. Wiesenfeld. Bifurcations in fluctuating systems: The centre manifold approach. *J Stat Phys*, 33:611–637, 1983.

- [95] Y. Kuramoto. Diffusion induced chaos in reactions systems. *Progr. Theoret. Phys. Suppl.*, 64:346–367, 1978.
- [96] Y.A. Kuznetsov. *Elements of applied bifurcation theory*, volume 112 of *Applied Mathematical Sciences*. Springer-Verlag, 1995.
- [97] P. Laure and Y. Demay. Symbolic computation and equation on the center manifold : application to the couette-taylor problem. *Comput. and Fluids*, 16(3):229–238, 1988.
- [98] T.K. Leen. A coordinate-independent center manifold reduction. *Phys. Lett. A*, 174(1-2):89–93, 1993.
- [99] G.X. Li and M.P. Paidoussis. Stability, double degeneracy and chaos in cantilevered pipes conveying fluid. *Internat. J. Non Linear Mech.*, 29(1):83–107, 1994.
- [100] X.D. Lin, J.W.H. So, and J.H. Wu. Centre manifolds for partial differential equations with delays. *Proc. Roy. Soc. Edinburgh Sect. A*, 122(3-4):237–254, 1992.
- [101] E. Lorenz and Krishnamurty. On the non-existence of a slow manifold. *J Atmos Sci*, 44:2940–2950, 1987.
- [102] E.N. Lorenz. On the existence of a slow manifold. *J Atmos Sci*, 43:1547–1557, 1986.
- [103] E.N. Lorenz. The slow manifold—what is it? *J. Atmospheric Sci.*, 49(24):2449–2451, 1992.
- [104] M. Luskin and G.R. Sell. Approximation theories for inertial manifolds. *Math Model & Num Anal*, 23:445–461, 1989.
- [105] M. Marion and R. Temam. Nonlinear galerkin methods. *SIAM J Numer Anal*, 26(5):1139–1157, 1989.
- [106] C. Martel and J.M. Vega. Finite size effects near the onset of the oscillatory instability. *Nonlinearity*, 9:1129–1171, 1996.
- [107] Z. Mei and A.J. Roberts. Equations for turbulent flood waves. In A. Mielke and K. Kirchgässner, editors, *Structure and dynamics of nonlinear waves in fluids*, pages 342–352. World Sci, 1995.
- [108] Z. Mei and A.J. Roberts. Modelling turbulent flood waves. In preparation, 1995.
- [109] G.N. Mercer and A.J. Roberts. A centre manifold description of contaminant dispersion in channels with varying flow properties. *SIAM J. Appl. Math.*, 50:1547–1565, 1990.
- [110] G.N. Mercer and A.J. Roberts. A complete model of shear dispersion in pipes. *Jap. J. Indust. Appl. Math.*, 11:499–521, 1994.
- [111] A. Mielke. *On Saint-venant's Problem And Saint-venant's Principle In Nonlinear Elasticity*. Trands In Appl Of Maths To Mech, 1988.
- [112] A. Mielke. On saint-venant's problem for an elastic strip. *Proc Roy Soc Edin A*, 110:161–181, 1988.
- [113] A. Mielke. Saint-venants problem and semi-inverse solutions in nonlinear elasticity. *Arch In Rat Mech & Anal*, 102:205–229, 1988.

- [114] A. Mielke. *Hamiltonian and Lagrangian flows on center manifolds*, volume 1489 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.
- [115] A. Mielke. Floquet theory for, and bifurcations from spatially periodic patterns. *Tatra Mt. Math. Publ.*, 4:153–158, 1994.
- [116] I.M. Moroz. Multiple instabilities in rotating convection. *Geophys. Astrophys. Fluid Dynamics*, 53(3):183–204, 1990.
- [117] R.G. Muncaster. Invariant manifolds in mechanics i: The general construction of coarse theories from fine theories. *Arch. Rat. Mech. Anal.*, 84:353–373, 1983.
- [118] R.G. Muncaster. Invariant manifolds in mechanics ii: Zero-dimensional elastic bodies with directors. *Arch. Rat. Mech. Anal.*, 84:375–392, 1983.
- [119] W. Nagata. A perturbed hopf bifurcation with reflection symmetry. *Proc. Roy. Soc. Edinburgh Sect. A*, 117(1-2):1–20, 1991.
- [120] N. Sri Namachchivaya and G. Leng. Equivalence of stochastic averaging and stochastic normal forms. *J Appl Mech*, 57:1011–1017, 1990.
- [121] N. Sri Namachchivaya and Y.K. Lin. Method of stochastic normal forms. *Int. J. Nonlinear Mechanics*, 26:931–943, 1991.
- [122] A.H. Nayfeh and S.A. Nayfeh. On nonlinear modes of continuous systems. *J Vibration Acoustics*, 116:129–136, 1994.
- [123] M.C. Neel. Convection in a horizontal porous layer of infinite extent. *European J. Mech. B Fluids*, 9(2):155–176, 1990.
- [124] M.C. Neel. Natural convection in a horizontal porous layer of infinite extent: inhomogeneous heating. In *European Conference on Iteration Theory (Batschuns, 1989)*, pages 272–278. World Sci. Publishing, River Edge, NJ, 1991.
- [125] M.C. Neel. Nonhomogeneous data in the choice of the convective configurations in a porous layer. *Nonlinear Vib. Probl.*, 25:297–311, 1993.
- [126] A.C. Newell, T. Passot, and J. Lega. Order parameter equations for patterns. *Annu. Rev. Fluid Mech.*, 25:399–453, 1993.
- [127] C. Nore and T.G. Shepherd. A hamiltonian weak-wave model for shallow water flow. *preprint*, 1996.
- [128] S.F. Nosov. The hamiltonian formalism in the drift theory of the motion of charged particles in a magnetic field. *Kinemat. Fiz. Nefesn. Tel*, 8(6):14–31, 1992.
- [129] D.H. Peregrine. Water waves and their development in space and time. *Proc. Roy. Soc. Lond. A*, 400:1–18, 1985.
- [130] D. Pfirsich and P.J. Morrison. The energy-momentum tensor for the linearized maxwell-vlasov and kinetic guiding center theories. *Phys. Fluids B*, 3(2):271–283, 1991.
- [131] P.K. Pollett and A.J. Roberts. A description of the long-term behaviour of absorbing continuous time markov chains using a centre manifold. *Advances Applied Probability*, 22:111–128, 1990.
- [132] M.R.E. Proctor. Planform selection by finite amplitude convection between poorly conducting slabs. *J. Fluid Mech.*, 113:469–485, 1981.

- [133] Th. Prokopiou, M. Cheng, and H.C. Chang. Long waves on inclined films at high reynolds number. *J. Fluid Mech.*, 222:665–691, 1991.
- [134] R.H. Rand and D. Armbruster. *Perturbation methods*. Applied Mathematical Sciences. Springer-Verlag, 1987.
- [135] R.H. Rand and W.L. Keith. Normal form and center manifold calculations on macsyma. In R. Pavelle, editor, *Applications of Computer Algebra*. Kluwer Acad, 1985.
- [136] A.J. Roberts. An analysis of near marginal, mildly penetrative convection with heat flux prescribed on the boundaries. *J. Fluid Mech.*, 158:71–93, 1985.
- [137] A.J. Roberts. Simple examples of the derivation of amplitude equations for systems of equations possessing bifurcations. *J. Austral. Math. Soc. B*, 27:48–65, 1985.
- [138] A.J. Roberts. The application of centre manifold theory to the evolution of systems which vary slowly in space. *J. Austral. Math. Soc. B*, 29:480–500, 1988.
- [139] A.J. Roberts. Appropriate initial conditions for asymptotic descriptions of the long term evolution of dynamical systems. *J. Austral. Math. Soc. B*, 31:48–75, 1989.
- [140] A.J. Roberts. The utility of an invariant manifold description of the evolution of a dynamical system. *SIAM J. Math. Anal.*, 20:1447–1458, 1989.
- [141] A.J. Roberts. Low-dimensionality—approximations in mechanics. In J. Noye and W. Hogarth, editors, *Computational Techniques and Applications: CTAC-89*, pages 715–722. Hemisphere, 1990.
- [142] A.J. Roberts. Boundary conditions for approximate differential equations. *J. Austral. Math. Soc. B*, 34:54–80, 1992.
- [143] A.J. Roberts. Planform evolution in convection—an embedded centre manifold. *J. Austral. Math. Soc. B*, 34:174–198, 1992.
- [144] A.J. Roberts. A sub-centre manifold description of the evolution and interaction of nonlinear dispersive waves. In L. Debnath, editor, *Nonlinear waves*, chapter 9, pages 127–156. World Sci, 1992.
- [145] A.J. Roberts. The invariant manifold of beam deformations. part 1:the simple circular rod. *J. Elas.*, 30:1–54, 1993.
- [146] A.J. Roberts. An accurate model of thin 2d fluid flows with inertia on curved surfaces. In P.A. Tyvand, editor, *Free-surface flows with viscosity*, Advances in Fluid Mechanics Series. Comput Mech Pub, 1996.
- [147] A.J. Roberts. Low-dimensional models of thin film fluid dynamics. *Phys. Letts. A*, 212:63–72, 1996.
- [148] A.J. Roberts. Low-dimensional modelling of dynamics via computer algebra. *Comput Phys Comm*, 100:215–230, 1997.
- [149] R. Valery Roy, A.J. Roberts, and M.E. Simpson. A lubrication model of coating flows over a curved substrate in space. preprint, USQ, August 1996.
- [150] R. Salmon. New equations for nearly geostrophic flow. *J. Fluid Mech.*, 153:461–477, 1985.

- [151] B. Scarpellini. Center manifolds of infinite dimensions. i. main results and applications. *Z. Angew. Math. Phys.*, 42(1):1–32, 1991.
- [152] G. Schöner and H. Haken. The slaving principle for stratonovich stochastic differential equations. *Z. Phys. B—Condensed matter*, 63:493–504, 1986.
- [153] G. Schöner and H. Haken. A systematic elimination procedure for ito stochastic differential equations and the adiabatic approximation. *Z. Phys. B—Condensed matter*, 68:89–103, 1987.
- [154] M. Schumaker. Center manifold reduction and normal form transformations in systems with additive noise. *Phys. Lett. A*, 122(6-7):317–322, 1987.
- [155] L.W. Schwartz and D.E. Weidner. Modeling of coating flows on curved surfaces. *J. Engrg Maths*, 29:91–103, 1995.
- [156] L.A. Segel. Distant side walls cause slow amplitude modulation of cellular convection. *J Fluid Mech*, 38:203–224, 1969.
- [157] S.W. Shaw. An invariant manifold approach to nonlinear normal modes of oscillation. *J. Nonlinear Sci.*, 4:419–448, 1994.
- [158] S.W. Shaw and C. Pierre. Normal modes for non-linear vibratory systems. *J Sound Vibration*, 164(1):85–124, 1993.
- [159] S.W. Shaw and C. Pierre. Normal modes of vibration for nonlinear continuous systems. *J Sound & Vib*, 169(3):319–347, 1994.
- [160] J. Sijbrand. Properties of centre manifolds. *Trans. Amer. Math. Soc.*, 289:431–469, 1985.
- [161] H. Sira-Ramirez. Sliding regimes on slow manifolds of systems with fast actuators. *Internat. J. Systems Sci.*, 19(6):875–887, 1988.
- [162] R. Smith. Longitudinal dispersion coefficients for varying channels. *J. Fluid Mech.*, 130:299–314, 1983.
- [163] R. Smith. Entry and exit conditions for flow reactors. *IMA J Appl Math*, 41:1–20, 1988.
- [164] A. Steindl. Hopf/steady-state mode interaction for a fluid conveying elastic tube with d_3 -symmetric support. In *Exploiting symmetry in applied and numerical analysis (Fort Collins, CO, 1992)*, volume 29 of *Lectures in Appl. Math.*, pages 373–387. Amer. Math. Soc., Providence, RI, 1993.
- [165] A. Steindl and H. Troger. Bifurcations of the equilibrium of a spherical double pendulum at a multiple eigenvalue. In *Bifurcation: analysis, algorithms, applications (Dortmund, 1986)*, volume 79 of *Internat. Schriftenreihe Numer. Math.*, pages 277–287. Birkhäuser, Basel-Boston, MA, 1987.
- [166] J. Swift and P.C. Hohenberg. Hydrodynamic fluctuations at the convective instability. *Phys. Rev. A*, 15:319–328, 1977.
- [167] P. Takac. Invariant 2-tori in the time-dependent ginzburg-landau equation. *Nonlinearity*, 5(2):289–321, 1992.
- [168] G.I. Taylor. Dispersion of soluble matter in solvent flowing slowly through a tube. *Proc. Roy. Soc. Lond. A*, 219:186–203, 1953.

- [169] G.I. Taylor. Conditions under which dispersion of a solute in a stream of solvent can be used to measure molecular diffusion. *Proc. Roy. Soc. Lond. A*, 225:473–477, 1954.
- [170] R. Temam. Do inertial manifold apply to chaos? *Physica D*, 37:146–152, 1989.
- [171] R. Temam. Inertial manifolds. *Mathematical Intelligencer*, 12:68–74, 1990.
- [172] E. Tirapegui. Stochastic processes in macroscopic physics. In *Connections among particle physics, nuclear physics, statistical physics and condensed matter (La Plata, 1987)*, pages 410–443. World Sci. Publishing, Teaneck, NJ, 1988.
- [173] E.S. Titi. On approximate inertial manifolds to the navier-stokes equations. *J. Math. Anal. & Appl.*, 149:540–557, 1990.
- [174] U.M. Titulaer. On the extraction of macroscopic descriptions from mesoscopic ones. In *Nonequilibrium statistical mechanics (Puerto Iguazu, 1989)*, pages 1–33. World Sci. Publishing, Teaneck, NJ, 1990.
- [175] N.G. van Kampen. Elimination of fast variables. *Physics Reports*, 124:69–160, 1985.
- [176] A. Vanderbauwhede. Centre manifolds. *Dynamics Reported*, pages 89–169, 1989.
- [177] A. Vanderbauwhede and G. Iooss. Center manifold theory in infinite dimensions. *Dynamics Reported*, 1:125–163, 1988.
- [178] S.D. Watt and A.J. Roberts. The accurate dynamic modelling of contaminant dispersion in channels. *SIAM J Appl Math*, 55(4):1016–1038, 1995.
- [179] R.B. White. Canonical hamiltonian guiding center variables. *Phys. Fluids B*, 2(4):845–847, 1990.
- [180] S. Wiggins. *Introduction to applied nonlinear dynamical systems and chaos*. Springer-Verlag, 1990.
- [181] A. Winfree. Patterns of phase compromise in biological cycles. *J. Math. Biol.*, 1:73–95, 1974.

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